NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 8

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Definition. Let k be a field. A k-algebra A is called *semi-simple* if every finitely generated left-A-module is semi-simple.

Exercise 1. Let k be a field and let A be a k-algebra. Prove the following statements.

(a) If A is a product of simple k-algebras, then A is semi-simple.
(b) Use the Double Centralizer Theorem and Exercise 2 of problem set 7 to show that every semi-simple k-algebra is a product of simple k-algebras.

Exercise 2. *(Remark: This is slightly tricky. If you wish, first accept this exercise as a black box and use it in Ex. 3)* Let k be a field of characteristic $\neq 2$ and $a, b \in k^\times$. Show that the following statements are equivalent.

(a) $\exists (x, y) \in k^2$ with $ax^2 + by^2 = 1$.
(b) $\exists (x, y, z) \in k^3 \setminus \{(0, 0, 0)\}$ with $ax^2 + by^2 = z^2$.
(c) $\exists (x, y, z, w) \in k^4 \setminus \{(0, 0, 0, 0)\}$ with $z^2 - ax^2 - by^2 + abw^2 = 0$.
(d) $\exists \gamma \in k(\sqrt{a})^\times$ with $b = N_{k(\sqrt{a})/k}(\gamma)$.

(Hint: For (c) $\Rightarrow$ (d), show that if $\sqrt{a} \notin k$, then $N(z+\sqrt{a}x) = bN(y+\sqrt{a}w)$ and conclude. For (b) $\Rightarrow$ (a), if $ax^2 + by^2 = 0$, one way to produce u, v with $au^2 + bv^2 = 1$ is to consider $[x : y : 0]$ as a point in the two dimensional projective space $\mathbb{P}^2_k$. Pick a line in $\mathbb{P}^2_k$ through $[x : y : 0]$, it will contain another solution to $aX^2 + bY^2 = Z^2$ (this is true for all lines through $[x : y : 0]$, except for precisely one, the so called "tangent to the curve defined by $aX^2 + bY^2 = Z^2$ at $[x : y : 0]$").)

Exercise 3. Let k be a field of characteristic $\neq 2$ and fix $a, b \in k^\times$. Last week, we defined the quaternion algebra $H(a, b; k)$. You proved that $H(a, b; k)$ is a central simple algebra over k and either a division algebra or isomorphic to $M_2(k)$. This exercise makes this destinction more precise.

(a) Show that if there are no elements $x, y \in k$ with $ax^2 + by^2 = 1$, then $H(a, b; k)$ is a division algebra. *(Hint: You can use (a)$\Leftrightarrow$(c) of the previous exercise.)*
(b) Show that if there exist $x, y \in k$ with $ax^2 + by^2 = 1$, then $H(a, b; k) \cong M_2(k)$ as k-algebras. In particular, $H(a, b; k)$ is not a division algebra. *(Hint: If $\sqrt{a} \in k$, then show that*
$$\alpha \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$
defines an isomorphism $H(a, b; k) \cong M_2(k)$, where $\alpha, \beta \in H(a, b; k)$ are such that $\alpha^2 = a$, $\beta^2 = b$, $\alpha\beta = -\beta\alpha$. If $\sqrt{a} \notin k$, write V for the 2-dimensional k-vector space $k(\sqrt{a})$, and let $\mu_z : V \to V$ be multiplication by $z \in k(\sqrt{a})$. Let $\sigma \in \mathrm{Gal}(k(\sqrt{a})/k)$ be the

If you want your solutions to be corrected, please hand them in just before the lecture on December 13, 2016. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.

non-trivial element and pick $\gamma \in k(\sqrt{a})$ *with* $N_{k(\sqrt{a})/k}(\gamma) = b$*. Then show that there is an isomorphism of* k*-algebras* $H(a,b;k) \xrightarrow{\cong} \mathrm{End}_k(V) \cong M_2(k)$ *satisfying*

$$1 \mapsto \mathrm{id}_V, \quad \alpha \mapsto \mu_\alpha, \quad \beta \mapsto \mu_\gamma \circ \sigma.)$$