NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 7

HÉLÈNE ESNAULT, LARS KINDLER

Exercise 1. Let k be an algebraically closed field.

(a) If D is a division algebra over k, show that $D = k$.
(b) If A is a k-algebra and M a simple left-A-module, then $\mathrm{End}_A(M) = k$.

Exercise 2. Let k be a field and let A be a k-algebra (finite dimensional over k by our conventions). Prove the following statements.

(a) If $L_1, \ldots, L_r$ and $N_1, \ldots, N_s$ are finitely generated left-A-modules, define $L := L_1 \oplus \ldots \oplus L_r$, $N := N_1 \oplus \ldots \oplus N_s$. Given an A-linear homomorphism $\varphi : L \to N$, we obtain for all pairs $(i, j) \in \{1, \ldots, s\} \times \{1, \ldots, r\}$ an A-linear homomorphism $\varphi_{ij} : L_j \to N_i$, by restricting φ to L_j and then composing with the projection $N \twoheadrightarrow N_i$.

Show that this construction induces an isomorphism of k-vector spaces

$$\mathrm{Hom}_A(L, N) \cong \bigoplus_{i,j} \mathrm{Hom}_A(L_j, N_i).$$

(b) If $L = L_1 \oplus \ldots \oplus L_r$, check that we can make

$$\bigoplus_{i,j=1}^{r} \mathrm{Hom}_A(L_j, L_i)$$

into a k-algebra by considering it as the k-vector space of $r \times r$-matrices where the entry with index (i, j) is an element of $\mathrm{Hom}_A(L_j, L_i)$ and where multiplication is matrix multiplication. More precisely: Given (φ_{ij}), (ψ_{ij}) in the direct sum, we have $\psi_{ij}, \varphi_{ij} : L_j \to L_i$, so it makes sense to define the product $(\gamma_{ij}) = (\varphi_{ij})(\psi_{ij})$ via $\gamma_{ij} = \sum_{k=1}^{r} \varphi_{ik}\psi_{kj} \in \mathrm{Hom}_A(L_j, L_i)$.

(c) With respect to this k-algebra structure, the isomorphism from part (a) of the exercise is in fact an isomorphism k-algebras

$$\mathrm{End}_A(L) \cong \bigoplus_{i,j=1}^{r} \mathrm{Hom}_A(L_j, L_i).$$

In particular, if $L_1 = L_2 = \ldots = L_r$, then it is an isomorphism

$$\mathrm{End}_A(L) \cong M_r(\mathrm{End}_A(L_1))$$

of k-algebras.

(d) Let V be a finite dimensional k-vector space and A a sub-k-algebra of $\mathrm{End}_k(V)$. This makes V into an A-module; show that the centralizer C of A in $\mathrm{End}_k(V)$ is $\mathrm{End}_A(V)$. Show that if V is semi-simple as an A-module, then C is a product of simple k-algebras.

If you want your solutions to be corrected, please hand them in on December 7, 2016. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.

Exercise 3. Let k be a field of characteristic $\neq 2$ and $a, b \in k^\times$.

(a) Define $A := k[X]/(X^2 - a)$, and $\alpha := X + (X^2 - a) \in A$. On A there is an involution $\overline{(-)}$ induced by $\bar{\alpha} := -\alpha$. Define $H(a, b; k) := A \times A$, and define multiplication on $H(a, b; k)$ as follows:
$$((x, y), (x', y')) \mapsto (xx' + by\overline{y'}, xy' + y\overline{x'}).$$
Show that this makes $H(a, b; k)$ into a k-algebra. It is called *generalized quarternion algebra.*

(b) Show that one can identify A with the subring $(x, 0) \in H(a, b; k)$, $x \in A$.

(c) Write $1 := (1, 0)$, $\alpha := (\alpha, 0)$ and $\beta := (0, 1)$. Show that $\alpha^2 = (a, 0)$, $\beta^2 = (b, 0)$ and $\alpha\beta = -\beta\alpha$ and that $1, \alpha, \beta, \alpha\beta$ is a basis of $H(a, b; k)$ as a k-vector space.

(d) If R is a 4-dimensional k-algebra with a k-basis $1, i, j, h \in R$ with $i^2 = a$, $j^2 = b$, $ij = -ji = h$, then show that $R \cong H(a, b; k)$.

(e) Show that $H(a, b; k)$ is a central simple algebra over k.

(f) Show that $H(a, b; k)$ is either a division algebra or isomorphic to $M_2(k)$.