NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 5

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Exercise 1. Let $(K, |\cdot|)$ be a complete discretely valued field. Fix an algebraic closure $\overline{K}$ of K (you may assume that K is perfect). We also denote by $|\cdot|$ the unique absolute value on $\overline{K}$ extending the one on K. On $K[T]$ we define the *Gauss norm*: For $\sum_{i=0}^{n} a_i T^i \in K[T]$, $\|f\| := \max_i |a_i|$.

(a) Given $f \in K[T]$ monic, show that $|\alpha| \leq \|f\|$ for every root $\alpha \in \overline{K}$ of f. (*Hint: First check that* $|\alpha| \leq \max_{i=1,\dots,n}\{|a_{n-i}|^{1/i}\}$.)

(b) Given $f, g \in K[T]$ monic polynomials of the same degree n, and $\alpha \in \overline{K}$ a root of f, show that

$$|g(\alpha)| \leq \|f - g\| \cdot \|f\|^{n-1}.$$

(c) Given $f, g \in K[T]$ monic polynomials of the same degree n, then for each root $\alpha \in \overline{K}$ of f there exists a root $\beta \in \overline{K}$ of g such that

$$|\alpha - \beta| \leq \|f - g\|^{1/n} \cdot \|f\|.$$

(d) *(Continuity of roots)* Let $f_i \in K[T]$, $i \in \mathbb{N}$, be a sequence of monic polynomials of the same degree n which converge to a polynomial $g \in K[T]$, with respect to the Gauss norm. For each i let $\alpha_i \in \overline{K}$ be a root of f_i. Then the sequence $(\alpha_i)_{i\in\mathbb{N}}$ contains a subsequence converging to a root of g.

If $\alpha \in \overline{K}$ is an element such that its minimal polynomial over K has at least two distinct roots, define $r(\alpha) := \min_{\gamma\neq\alpha} |\alpha - \gamma| > 0$, where γ runs through the conjugates of α.

(e) *(Krasner's Lemma)* Assume that $\alpha \in \overline{K}$ is separable of degree > 1 over K. Show that if $\beta \in \overline{K}$ is such that $|\alpha - \beta| < r(\alpha)$, then $K(\alpha) \subseteq K(\beta) \subseteq \overline{K}$.

(f) Let $f \in K[T]$ be an irreducible, monic, separable polynomial of degree > 1. Then any monic polynomial $g \in K[T]$ with $\deg(g) = \deg(f)$ which is sufficiently close to f is also irreducible and separable. For each root β of g there exists a root α of f such that $|\beta - \alpha| < r(\alpha)$. For such α, β we have $K(\alpha) = K(\beta)$.

Exercise 2. Let p be a prime number and fix an algebraic closure $\overline{\mathbb{Q}_p}$ of the field of p-adic numbers. From the lecture you know that the p-adic absolute value extends uniquely to $\overline{\mathbb{Q}_p}$.

(a) Show that for every finite extension $K/\mathbb{Q}_p$ there exists a finite extension $L/\mathbb{Q}$ contained in K such that $[L : \mathbb{Q}] = [K : \mathbb{Q}_p]$ and such that $K = L\mathbb{Q}_p$.

(b) Denote the the completion of $\overline{\mathbb{Q}_p}$ by $\mathbb{C}_p$. Show that $\mathbb{C}_p$ is algebraically closed.

Exercise 3. Let $(K, |\cdot|)$ be a complete discretely valued field. Let $\mathcal{O}_K$ be its ring of integers and $\mathfrak{m}$ its maximal ideal.

If you want your solutions to be corrected, please hand them in just before the lecture on November 22, 2016. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.

(a) $\mathcal{O}_K$ equipped with the subspace topology is compact if and only if the residue field $\mathcal{O}_K/\mathfrak{m}$ is finite.
(b) Conclude that if $\mathcal{O}_K$ is compact then K is locally compact (i.e. every element of K has a compact neighborhood).[1]
(c) Conclude also that if K has a finite residue field, then for all n, the sets $\mathfrak{m}^n$, $1+\mathfrak{m}^n$, $\mathcal{O}_K^\times$ are compact.
(d) Fix an algebraic closure $\overline{K}$ of K. If K has characteristic 0 and if K has a finite residue field, then for every $n \in \mathbb{N}$, there are only finitely many extensions of K of degree $\leq n$ contained in $\overline{K}$.

[1]A field K with nontrivial absolute value $|\cdot|$ is called *local field* if K is locally compact with respect to the topology induced by $|\cdot|$.