NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 4

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Exercise 1. (a) Let K be a field of characteristic $p > 0$ and let L/K be a purely inseparable algebraic extension. This means that for every $x \in L$, there exists some $n > 0$ such that $x^{p^n} \in K$. Show that if $|\cdot|$ is an absolute value on K, then $|\cdot|$ extends uniquely to L.

(b) Conclude that if $(K, |\cdot|)$ is complete, then $|\cdot|$ extends uniquely to any algebraic extension of K (in the lecture we only saw this in case L/K is separable).

(c) Let $(K, |\cdot|)$ be a complete discretely valued field. Show that the unique extension of $|\cdot|$ to any algebraic closure of K is not discrete.

Exercise 2. Let K be a field and $|\cdot|$ a nontrivial absolute value on K. Denote by $\widehat{K}$ the completion of K with respect to $|\cdot|$ and fix an algebraic closure $\overline{\widehat{K}}$. We also denote by $|\cdot|$ the unique absolute value on $\overline{\widehat{K}}$ extending the absolute value[1] of $\widehat{K}$.

Let $L := K(\alpha)$ be a finite separable extension and let $f \in K[T]$ be the (monic) minimal polynomial of α. Write $\hat{f}$ for f considered as an element of $\widehat{K}[T]$.

(a) Let $\sigma : L \hookrightarrow \overline{\widehat{K}}$ be a K-linear embedding. This defines an absolute value on L extending the absolute value on K: For $y \in L$, set $|y|_\sigma := |\sigma(y)|$. The embedding σ is given by mapping α to a root of $\hat{f}$, thus there exists a unique monic irreducible factor $f_\sigma \in \widehat{K}[T]$ of $\hat{f}$ such that $f_\sigma(\sigma(\alpha)) = 0$. Show that if $\tau : L \hookrightarrow \overline{\widehat{K}}$ is a second K-linear embedding, then $|\cdot|_\sigma = |\cdot|_\tau$ if $f_\sigma = f_\tau$.

Conclude that this construction defines a map from the set of irreducible components of $\hat{f}$ to the set of absolute values on L extending $|\cdot|$.

(b) If $|\cdot|_L$ is an absolute value on L extending $|\cdot|$, denote by $\widehat{L}$ the completion of L with respect to $|\cdot|_L$. Consider the diagram

$$\begin{tikzcd} L = K(\alpha) \arrow[r, hook] & \widehat{L} \\ K \arrow[u, hook] \arrow[r, hook] & \widehat{K}. \arrow[u, hook] \end{tikzcd}$$

Using this diagram, we consider α as an element of $\widehat{L}$ algebraic over $\widehat{K}$. Let $h \in \widehat{K}[T]$ be its minimal polynomial. Show that h is an irreducible factor of $\hat{f}$ and that this construction induces a map from the set of equivalence classes of absolute values on L extending $|\cdot|$ to the set of irreducible factors of $\hat{f}$.

If you want your solutions to be corrected, please hand them in just before the lecture on November 15, 2016. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.

[1] You can use that such a unique extension exists, even though in the lecture we only proved it in case $|\cdot|$ is nonarchimedean

(c) Show that the constructions from the previous two exercises are mutually inverse, i.e., that they give a bijection

$$\{\text{monic irreducible factors of } \hat{f}\} \leftrightarrow \{\text{equiv. classes of absolute values on } L \text{ extending } |\cdot|\}$$

Exercise 3. Let K be a field equipped with an absolute value $|\cdot|$. Let L/K be a finite separable extension.

(a) Show that there are a finite number of absolute values $|\cdot|_1, \ldots, |\cdot|_g$ on L extending $|\cdot|$, such that any absolute value on L extending $|\cdot|$ is equivalent to one of the $|\cdot|_1, \ldots, |\cdot|_g$.

(b) For $i \in \{1, \ldots, g\}$ write L_i for the completion of L with respect to $|\cdot|_i$. We obtain a diagram

$$\begin{tikzcd} L = K(\alpha) \arrow[r, hook] & L_i \\ K \arrow[u, hook] \arrow[r, hook] & \widehat{K}. \arrow[u, hook] \end{tikzcd}$$

Show that the map

$$\begin{aligned} L \otimes_K \widehat{K} &\longrightarrow \textstyle\prod_{i=1}^{g} L_i \\ a \otimes b &\longmapsto (ab)_i \end{aligned}$$

is an isomorphism of K-algebras.

Exercise 4. We generalize Ostrowski's theorem to number fields. Let K be a number field. Then every nontrivial absolute value on K is equivalent to one and only one of the following:

- For a nonzero prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$ and $x \in K$, define $|x|_{\mathfrak{p}} := 1/\mathbb{N}(\mathfrak{p})^{v_{\mathfrak{p}}(x)}$, where $v_{\mathfrak{p}}(x) = \max\{n \in \mathbb{Z} | x \in \mathfrak{p}^n\}$ and $\mathbb{N}(\mathfrak{p}) = p^f$, if p is a prime such that $(p) = \mathbb{Z} \cap \mathfrak{p}$ and $f := [\mathcal{O}_K/\mathfrak{p} : \mathbb{F}_p]$.
- If $\sigma_1, \ldots, \sigma_r$ are the real embeddings of K, then for $x \in K$ and $i = 1, \ldots, r$ define $|x|_{\sigma_i} = |\sigma_i(x)|$, where the right-hand side is the usual archimedean absolute value on $\mathbb{R}$.
- If $\sigma_{r+1}, \bar{\sigma}_{r+1}, \ldots, \sigma_{r+s}, \bar{\sigma}_{r+s}$ are the complex embeddings of K, then for $x \in K$ define $|x|_{\sigma_i} = |\sigma_i(x)|$ for $i = r+1, \ldots, r+s$. (Note that $|\sigma_i(x)| = |\bar{\sigma}_i(x)|$.)

You can proceed as follows.

(a) If $|\cdot|$ is a nonarchimedean absolute value on K, let A be its valuation ring and $\mathfrak{m} \subseteq A$ its maximal ideal.
  (i) Show that $\mathcal{O}_K \subseteq A$.
  (ii) If $|\cdot|$ is nontrivial, then $\mathfrak{m} \cap \mathcal{O}_K =: \mathfrak{p}$ is a nonzero prime ideal.
  (iii) If $\mathcal{O}_{K,\mathfrak{p}}$ denotes the localization of $\mathcal{O}_K$ at $\mathfrak{p}$, show that $\mathcal{O}_{K,\mathfrak{p}} \subseteq A$ and $\mathfrak{p}\mathcal{O}_{K,\mathfrak{p}} = \mathcal{O}_{K,\mathfrak{p}} \cap \mathfrak{m}$.
  (iv) Conclude that $|\cdot|$ and $|\cdot|_{\mathfrak{p}}$ equivalent.

(b) Assume that $|\cdot|$ is archimedean.
  (i) Show that without loss of generality we may assume that the restriction of $|\cdot|$ to $\mathbb{Q}$ is the usual archimedean absolute value $|\cdot|_\infty$ (*Hint: By Ostrowski's theorem* $|x| = |x|_\infty^t$ *for some* $t \in \mathbb{R}_{>0}$ *and all* $x \in \mathbb{Q}$. *Deduce that* $|\cdot|^{1/t}$ *is an absolute value on* K).
  (ii) Write $K = \mathbb{Q}(\alpha)$, let $f \in \mathbb{Q}[T]$ be the minimal polynomial of α and relate the archimedean absolute values on K to the irreducible factors of $\hat{f} \in \mathbb{R}[T]$.