NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 13

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Exercise 1. Define $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and consider $\mathbb{T}$ as a topological group with respect to addition. For a topological group G, write $G^* := \mathrm{Hom}_{\cong}(G, \mathbb{T})$ and equip it with the compact-open topology. Prove the following statements.

(a) Every proper closed subgroup of $\mathbb{T}$ is finite.
(b) If G is a profinite group, show that every continuous homomorphism $G \to \mathbb{T}$ has finite image.

Disclaimer: The following exercises are perhaps a bit long and meant to present some cohomological aspects of the theory of Brauer groups. If they are too hard or too long, feel free to look them up in a book.

Exercise 2. Let L/K be a finite Galois extension with Galois group G. A map $\varphi : G \times G \to L^\times$ is called *2-cocycle*, if for all $\rho, \sigma, \tau \in G$, the so called cocycle condition is fulfilled, i.e., we have

$$\rho(\varphi(\sigma, \tau)) \cdot \varphi(\rho, \sigma\tau) = \varphi(\rho\sigma, \tau) \cdot \varphi(\rho, \sigma).$$

(a) Write $A := \bigoplus_{\sigma \in G} e_\sigma L$ as an L-vector space, and if $\varphi : G^2 \to L^\times$ is a 2-cocycle, define a multiplication $A \times A \to A$ by linearly extending

$$(\alpha e_\sigma) \cdot (\beta e_\tau) := \alpha\sigma(\beta)\varphi(\sigma, \tau)e_{\sigma\tau}$$

for $\alpha, \beta \in L$, $\sigma, \tau \in G$. Show that A is a central simple algebra over K. We denote it by $A(\varphi)$.

(b) Show that $L \cong L\varphi(\mathrm{id}, \mathrm{id})^{-1}e_{\mathrm{id}} \subseteq A$ is a subfield splitting $A(\varphi)$.

(c) Two 2-cocycles $\psi, \varphi : G^2 \to L^\times$ are said to be equivalent if there exists a map $\theta : G \to L^\times$ such that

$$\psi(\sigma, \tau)\varphi(\sigma, \tau)^{-1} = \theta(\sigma) \cdot \sigma(\theta(\tau)) \cdot \theta(\sigma\tau)^{-1}.$$

Show that $A(\psi) \cong A(\varphi)$ if and only if φ and ψ are equivalent.

(d) Define $H^2(G, L^\times)$ to be the group of 2-cocycles modulo the equivalence relation defined above. Show that $H^2(G, L^\times)$ is a group and that the assignment $[\varphi] \mapsto [A(\varphi)]$ gives rise to an injective map $H^2(G, L^\times) \to \mathrm{Br}(K)$. (This is in fact an isomorphism of groups.).

(e) If $\varphi, \psi : G^2 \to L^\times$ are two 2-cocycles, then $[A(\varphi) \otimes A(\psi)] = [A(\varphi \cdot \psi)] \in \mathrm{Br}(L/K)$. We have seen the strategey of the proof before. Proceed as follows:

(i) Write $A = A(\varphi)$, $B = A(\psi)$, $C = A(\varphi \cdot \psi)$, and let $V = A^{\mathrm{op}} \otimes_L B$. For $x \in L$ we then have $x(a \otimes_L b) = (xa) \otimes_L b = a \otimes_L (xb)$.

(ii) Let $A \otimes_K B$ act on V from the right via $(a \otimes_L b)(a' \otimes_K b') := aa' \otimes_L bb'$.

If you want your solutions to be corrected, please hand them in just before the lecture on February 3, 2017. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.

(iii) Let C act on V from the left as follows. Let $\{u_\sigma\}$, $\{v_\sigma\}$, $\{w_\sigma\}$ be bases of A, B, C as in part (a). Then for $a \in A, b \in B$, $x \in L, \sigma \in G$ define:

$$(xw_\sigma)(a \otimes_L b) := xu_\sigma a \otimes_L v_\sigma b.$$

Check that this defines a left-C-action on V which is compatible with the right $A \otimes_K B$-action, i.e., such that we obtain a homomorphism $f : (A \otimes_K B)^{\mathrm{op}} \to \mathrm{End}_C(V)$.

(iv) Show that f is an isomorphism and conclude that $[A \otimes_K B] = [C]$ in $\mathrm{Br}(K)$.

Exercise 3. Let K be a field and fix a separable closure $\overline{K}$. If $\chi : \mathrm{Gal}(\overline{K}/K) \to \mathbb{Q}/\mathbb{Z}$ is a continuous homomorphism, we saw that χ corresponds to a finite cyclic extension L/K (the fixed field of $\ker(\chi)$) together with the choice of a generator σ of the cyclic group $\mathrm{Gal}(L/K)$. Write $n = [L : K] = \mathrm{ord}(\chi)$. Define the map $\varphi_{\chi,a} : \mathrm{Gal}(L/K)^2 \to L^\times$ as

$$\varphi_{\chi,a}(\sigma^i, \sigma^j) := \begin{cases} 1 & \text{if } 0 \leq i, j, i+j < n \\ a & \text{if } 0 \leq i, j < n \leq i+j \end{cases}$$

(a) Show that $\varphi_{\chi,a}$ is a 2-cocycle.

(b) Show that $A(\chi, a) \cong A(\varphi_{\chi,a})$.