Prof. Dr. Konrad Polthier
Dr. Tillmann Kleiner
AG Mathematical Geometry Processing

Differential Geometry II
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Exercise Sheet 2

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Submission: 4.5.2026, 8:30 PM (start of the tutorial) or 10:15 PM (start of the lecture)

Exercise 1. (4 points)
Prove the following statements:

a) For two differentiable vector fields X and Y we define $XY(f) \coloneqq X(Y(f))$ for every differentiable function $f\colon M \to \mathbb{R}$. Find two differentiable vector fields X and Y on $\mathbb{R}^2$ such that XY is *not* a differentiable vector field.

b) Show that the Lie bracket $[X, Y]$ of two differentiable vector fields X and Y is again a differentiable vector field.

Exercise 2. (4 points)
Let $U, V, W \in TM$ be differentiable vector fields on a k-manifold M, let $f, h : M \to \mathbb{R}$ be differentiable functions and $\alpha, \beta \in \mathbb{R}$. Prove the following properties of the Lie bracket:

a) $[\alpha U + \beta V, W] = \alpha\,[U, W] + \beta\,[V, W]$,

b) $[U, V] = -\,[V, U]$,

c) $[fU, hV] = f \cdot h \cdot [U, V] + f \cdot U(h) \cdot V - h \cdot V(f) \cdot U$,

d) $[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0$.

Exercise 3. (4 Points)
Consider $\mathbb{R}^2$ with the standard coordinates denoted by x and y and define the vector fields

$$W_1 = X \coloneqq x\frac{\partial}{\partial x}, \qquad W_2 = Y \coloneqq y\frac{\partial}{\partial y}, \qquad W_3 = U \coloneqq x\frac{\partial}{\partial y} \qquad \text{and} \qquad W_4 = V \coloneqq y\frac{\partial}{\partial x}.$$

Calculate the matrix of vector fields $([W_i, W_j])_{i,j=1,2,3,4}$.

Exercise 4. (4 points)
Consider the vector product $\times$ on $\mathbb{R}^3$ given by

$$x \times y \coloneqq \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} \quad \text{for } x, y \in \mathbb{R}^3.$$

Write down the algebraic properties of the binary operation $[x, y] \coloneqq x \times y$ on $\mathbb{R}^3$ that are analogous to the Lie bracket. Also, state the source where you looked up the property and a name for the property.