

Exercise Sheet 9

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Submission: 5.1.2026, 8:30 AM (start of the tutorial) or 10:15 AM (start of the lecture)

Exercise 1. (4 points)

Consider a parametrized surface $M := f(\Omega)$, $f: \Omega \rightarrow \mathbb{R}^3$ with negative Gaussian curvature $K < 0$ everywhere. Prove the following statements:

- i) At each point $p \in M$ there are two linearly independent asymptotic directions, i.e. two linearly independent tangent vectors $X, Y \in T_p M$ such that $b(X, X) = b(Y, Y) = 0$.¹
- ii) The surface M is a minimal surface if and only if the asymptotic directions in each point on M are perpendicular to each other.

Exercise 2. (4 points)

Let p be a point on a surface M . Show that the mean curvature at p is given by

$$H(p) := \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\varphi) d\varphi, \quad (1)$$

where $\kappa_n(\varphi)$ denotes the normal curvature at p in the direction v_p spanning a fixed oriented angle of φ to a fixed reference direction $v_0 \in T_p M$.

Exercise 3. (4 points)

Consider the parametrized surface $f: \Omega \rightarrow \mathbb{R}^3$ with

$$f(u, v) = \left(u, v, \log \left(\frac{\cos(v)}{\cos(u)} \right) \right) \quad \Omega := \left\{ (u, v) \mid \cos(u) \neq 0 \text{ and } \frac{\cos(v)}{\cos(u)} > 0 \right\}. \quad (2)$$

- i) Give a more explicit description of the domain Ω .
- ii) Prove that f defines a minimal surface.
- iii) Sketch the surface for $(u, v) \in \Omega \cap (-\pi/2, 3\pi/2) \times (-\pi/2, 3\pi/2)$.

Exercise 4. (4 points)

Calculate the conformally parametrized minimal surface from the Weierstraß representation with the functions $F(z) = z^2$ and $G(z) = 1/z$. Compare the resulting surface with the example from the lecture, that used $F(z) = 1$ and $G(z) = z$.

Exercise 5. (4 bonus points)

Let $f: \Omega \rightarrow \mathbb{R}^3$ be a regular parametrization of a surface $M := f(\Omega)$ and $u \in \Omega$ and $p = f(u) \in M$. Define the operators

$$\hat{S}_{|p}: T_p M \longrightarrow T_p M \quad X \longmapsto \hat{S}_{|p}(X) := Df_{|u}(S_{|u}((Df_{|u})^{-1}(X))), \quad (3a)$$

$$\hat{b}_{|p}: T_p M \times T_p M \longrightarrow \mathbb{R}, \quad (X, Y) \longmapsto \hat{b}_{|p}(X, Y) := b_{|u}((Df_{|u})^{-1}(X), (Df_{|u})^{-1}(Y)), \quad (3b)$$

where $(Df_{|u})^{-1}$ denotes the inverse of the differential of f at u , understood as a bijective linear mapping $Df_{|u}: T_u \Omega \rightarrow T_p M$. Further, $S_{|u}$ denotes the shape operator and $b_{|u}$ the second fundamental form at u , as obtained from the parametrization f . Prove, that the definitions of $\hat{S}_{|p}$ and $\hat{b}_{|p}$ are invariant with respect to reparametrizations.²

Exercise 6. (0 points)

Enjoy your holidays and have a good start into the new year!

¹For $X, Y \in T_p M$ the expression $b(X, Y)$ is a short-hand for $b(Df_{|u}(v), Df_{|u}(w))$ where $u \in \Omega$ with $p = f(u)$ and $v, w \in T_u \Omega$ such that $X = Df_{|u}(v)$ and $Y = Df_{|u}(w)$. With Exercise 5 on this exercise sheet it is proved that this is well-defined, and thus, a consistent convention.

²That is, we get the same operators $\hat{S}_{|p}$ and $\hat{b}_{|p}$ for another parametrization $\tilde{f}: \tilde{\Omega} \rightarrow \mathbb{R}^3$ related to f by a diffeomorphic reparametrization, that is, a diffeomorphism $\varphi: \Omega \rightarrow \tilde{\Omega}$ such that $f = \tilde{f} \circ \varphi$.