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AG Mathematical Geometry Processing

Differential Geometry I
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Freie Universität Berlin

Exercise Sheet 8

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Submission: 15.12.2025, 8:30 AM (start of the tutorial) or 10:15 AM (start of the lecture)

Exercise 1. (8 points)
Let $f\colon \Omega \to \mathbb{R}^3$ be a regular surface parametrization and let $c = f \circ \gamma\colon I \to f(\Omega)$ be a regular arc-length parametrized curve on $f(\Omega)$.

i) Prove, using the definition via the Darboux frame, that the geodesic curvature of c is given by

$$\kappa_{\mathrm{g}} = \langle c'', N \times c' \rangle. \tag{1}$$

How does this relate to the oriented curvature $\kappa_{\mathrm{orient}} = \langle \tilde{c}'', J\tilde{c}' \rangle$ of planar curves $\tilde{c}$?

ii) Define the matrix $(J_{ij})_{i,j} := (\langle f_{u^i}, N \times f_{u^j} \rangle)_{i,j}$ with the partial derivatives $f_{u^i} = \partial f / \partial u^i$ and the surface normal $N = f_{u^1} \times f_{u^2} / |f_{u^1} \times f_{u^2}|$ and prove

$$(J_{ij})_{i,j} = \sqrt{\det g} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

iii) Prove, using (1) and the notation from Part ii), that the geodesic curvature of c satisfies

$$\kappa_{\mathrm{g}} = \sum_{k,l=1,2} J_{kl} \left(\gamma_k'' + \sum_{i,j=1,2} \Gamma_{ij}^k \gamma_i' \gamma_j' \right) \gamma_l'. \tag{3}$$

Exercise 2. (8 points)
Consider the two-dimensional hyperbolic space $\mathbb{H}^2 = \{(u,v) \in \mathbb{R}^2 \mid v > 0\}$ with the metric $g = \frac{1}{v^2} I_2$ for $(u,v) \in \mathbb{H}^2$ from Exercise 4 on Sheet 6. The circles C_{h} in $\{(u,v) \in \mathbb{R}^2 \mid v \geq 0\}$ with radius $r > 0$ that are tangential to the line $\{(u,0) \mid u \in \mathbb{R}\}$ (the "boundary" of $\mathbb{H}^2$) are called *horocycles*.

i) Prove, using Equations (2) and (3), that the horocycle $(-\pi, \pi) \ni t \mapsto (-\sin(t), 1 + \cos(t))$ in $\mathbb{H}^2$ has constant geodesic curvature $\kappa_{\mathrm{g}} = 1$ with respect to the metric g.

ii) Prove, that the metric g on $\mathbb{H}^2$ is invariant with respect to $(u,v) \mapsto (au+b, av)$ for $a > 0$ and $b \in \mathbb{R}$. Conclude that *all* horocycles in $\mathbb{H}^2$ have constant absolute geodesic curvature $|\kappa_{\mathrm{g}}| = 1$.

iii) Determine $\alpha > 0$ such that the segment $[0, \alpha] \ni t \mapsto (-\sin(t), 1 + \cos(t))$ of the horocycle from Part i) has the length π with respect to the metric g on $\mathbb{H}^2$.

Exercise 3. (4 bonus points)
Verify, that there exists a regular surface parametrization $f\colon \Omega = \mathbb{R} \times (-\pi, \pi) \to \mathbb{R}^3$ with the following first and second fundamental forms:

$$(g_{ij})|_{(u,v)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad (b_{ij})|_{(u,v)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for } (u,v) \in \Omega.$$

Determine the corresponding surface parametrization with the initial values

$$f(0,0) = (1,0,0), \qquad f_u(0,0) = (0,0,1), \qquad f_v(0,0) = (0,1,0)$$

and describe its shape.

Hint: Integrate curves $c = f \circ \gamma$ along parameter lines $u = const$ and $v = const$ using the Darboux equations. Then combine the results.