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Version: 1

Differential Geometry II
Summer Semester 2024
Freie Universität Berlin

Exercise Sheet 4

Submission: 28.05.2024, 12:15 PM (start of lecture)

Exercise 1. (5 points)
Let X, U, V, W be vector fields on a Riemannian manifold. Show that the *Riemannian curvature tensor* R fulfills the following properties:

i) $R(U,V)W + R(V,W)U + R(W,U)V = 0$,

ii) $(\nabla_U R)(V,W)X + (\nabla_V R)(W,U)X + (\nabla_W R)(U,V)X = 0$,

iii) $g(R(U,V)W,X) = -g(R(U,V)X,W)$, and

iv) $g(R(U,V)W,X) = g(R(W,X)U,V)$.

Exercise 2. (3 points)
Let ∇ and $\tilde{\nabla}$ be two connections on a manifold M. Show that $\nabla - \tilde{\nabla}$ is a $(1,2)$-tensor field on M. Show that the assignment $(X,Y) \mapsto \nabla_X Y$ is not a $(1,2)$-tensor for vector fields X, Y.

Exercise 3. (4 points)
Let (M,g) be a Riemannian manifold with normal coordinates (U, x_i) around $p \in M$ (see previous exercise sheet). For $W = \sum_{i=0}^n W_i \partial_i \in T_pM$ show that the Jacobi field along a radial geodesic γ (i.e. $\gamma(0) = p$) with $J(0) = 0$ and $J'(0) = W$ is given by $J(t) = t \sum_{i=1}^n W_i \partial_i$ for all t.