Prof. Dr. Konrad Polthier
Dr. Eric Zimmermann
Version: 1

Differential Geometry II
Summer Semester 2024
Freie Universität Berlin

Exercise Sheet 3

Submission: 21.05.2024, 12:15 PM (start of lecture)

Exercise 1. (5 points)
Let (M, g) be a Riemannian manifold, $p \in M$ and $E_1, \dots, E_n$ be an orthonormal basis of T_pM. This basis induces an isomorphism $E : \mathbb{R}^n \to T_pM$, $(x_1, \dots, x_n) \mapsto \sum_{i=1}^n x_i E_i$. On a (sufficiently) small neighborhood U of p where $\exp_p$ is bijective, $x \coloneqq E^{-1} \circ \exp_p^{-1} : U \to \mathbb{R}^n$ is called a *normal coordinate system* for M at p. Show:

i) The coordinates of p are $(0, \dots, 0)$ and $g = (\delta_{ij})$ at p.

ii) For any $V = \sum_{i=1}^n V_i \partial_i \in T_pM$, the geodesic c emanating from p with initial velocity V is given by $x(c(t)) = (tV_1, \dots, tV_n)$.

iii) The first partial derivatives of g_{ij} and the Christoffel symbols vanish at p.

Exercise 2. (4 points)
Let (M, g) be a Riemannian manifold with Levi Cività connection ∇. Let R be the $(1,3)$-curvature tensor

$$R(X, Y)Z \coloneqq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

i) Show that R is tensorial in Z, i.e. $R(X, Y)(fZ) = fR(X, Y)Z$ for a C^∞ function f.

ii) Determine the coordinates R^l_{kij} of R in the coordinate expression $R(\partial_i, \partial_j)\partial_k = \sum_l R^l_{kij} \partial_l$.

Exercise 3. (7 points)
Consider the open unit disk in $\mathbb{R}^2$ given in polar coordinates $\{(r, \varphi) \in [0, 1[\times [0, 2\pi[\}$ with the following metric

$$g = \frac{4}{(1 - r^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

i) Sketch ∂_r and ∂_φ and determine $|\partial_r|$ and $|\partial_\varphi|$.

ii) Determine $\nabla_{\partial_r} \partial_r$, $\nabla_{\partial_r} \partial_\varphi$, $\nabla_{\partial_\varphi} \partial_r$, $\nabla_{\partial_\varphi} \partial_\varphi$, and $\nabla_V W$ for $V = r\partial_r + r^2 \partial_\varphi$ and $W = \varphi \partial_r + r\varphi \partial_\varphi$. Why do two of these derivatives coincide?