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Version: 1

Differential Geometry II
Summer Semester 2024
Freie Universität Berlin

Exercise Sheet 2

Submission: 07.05.2024, 12:15 PM (start of lecture)

Exercise 1. (5 points)
Show the following statements:

i) For two derivations X and Y and a smooth function we define $XY(f) := X(Y(f))$. Find two derivations X and Y on $\mathbb{R}^2$ such that XY is *not* a derivation.

ii) Show that the Lie bracket $[X, Y]$ of two vector fields is again a derivation at each point.

Exercise 2. (4 points)
Let $U, V, W \in TM$ be vector fields on a k-manifold M, let $f, h : M \to \mathbb{R}$ be differentiable functions and $\alpha, \beta \in \mathbb{R}$. Show the following properties of the Lie bracket:

i) $[\alpha U + \beta V, W] = \alpha\,[U, W] + \beta\,[V, W]$,

ii) $[U, V] = -\,[V, U]$,

iii) $[fU, hV] = f \cdot h \cdot [U, V] + f \cdot U(h) \cdot V - h \cdot V(f) \cdot U$,

iv) $[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0$.

Exercise 3. (6 points)
Consider $\mathbb{R}^n$ with the standard metric. Show that the directional derivative D given at $p \in \mathbb{R}^n$ by

$$D_V W_{|_p} := \lim_{t \to 0} \frac{W_{|_{p+tV}} - W_{|_p}}{t} = DW \cdot V_{|_p}$$

is a Riemannian connection.