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Version: 1

Differential Geometry I
Winter Semester 2023/2024
Freie Universität Berlin

Exercise Sheet 10

Submission: 23.01.2024, 12:15 PM (start of lecture)

Exercise 1. (4 points)
Let p be a point on a surface. Show that the mean curvature at p is given by

$$H(p) := \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\varphi) d\varphi,$$

where $\kappa_n(\varphi)$ denotes the normal curvature at p in the direction v_p spanning an angle of φ to a fixed reference direction $v_0 \in T_p f$.

Exercise 2. (9 points)
Let $z_0 = 0$ and $F, G : \mathbb{C} \to \mathbb{C}$ be given by $F(z) = 1$ and $G(z) = z$.

i) Use the Weierstraß representation to compute from F and G as given above a parametrization of the corresponding surface $f : \mathbb{R}^2 \to \mathbb{R}^3$, $(u, v) \mapsto (f_1(u, v), f_2(u, v), f_3(u, v))$.[1]

ii) Show that all surfaces of the *associate family* $\left(e^{i\varphi} F, G\right)$, $\varphi \in [0, 2\pi)$, are intrinsically isometric. Plot the corresponding members of the associate family for $\varphi \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\right\}$.

iii) For $k \in \{1, 2, 3\}$, plot f on disk-like[2] domains $D_k \subset \mathbb{R}^2$ such that f is

   a) embedded,
   b) nearly intersecting,
   c) a very large disk.

iv) Determine a parametrization using the Weierstraß representation from $F(z) = z^2$, $G(z) = z^{-1}$. What can you observe comparing it with the one obtained in i)?

Exercise 3. (3 points)
Calculate the holomorphic functions $\varphi_1, \varphi_2, \varphi_3$ of the surface

$$f(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, -v + \frac{1}{3}v^3 - vu^2, u^2 - v^2\right)$$

and verify $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$.

[1] This surface is called *Enneper surface*.
[2] Disk-like includes rectangular domains.