Prof. Dr. Konrad Polthier
Eric Zimmermann
Version: 1

Differential Geometry I
Winter Semester 2023/2024
Freie Universität Berlin

Exercise Sheet 4

Submission: 21.11.2023, 12:15 PM (start of lecture)

Note: This sheet contains 3 bonus points.

Exercise 1. (4 points)
The one parameter family of surfaces

$$f : [0, 2\pi] \times (-\infty, \infty) \times [0, \pi] \to \mathbb{R}^3$$
$$(u, v, t) \mapsto \begin{pmatrix} \cos(t)\cos(u)\cosh(v) + \sin(t)\sin(u)\sinh(v) \\ -\cos(t)\sin(u)\cosh(v) + \sin(t)\cos(u)\sinh(v) \\ \cos(t)v + \sin(t)u \end{pmatrix}$$

describes a transformation of the *catenoid* $f(-;-;0)$ into the *helicoid* $f(-;-;\pi/2)$. Show that this transformation has the following properties:

i) The surface normals remain unchanged, i.e. $\frac{\partial}{\partial t}N = 0$;

ii) All surfaces $f(-;-;t)$ are isometric, i.e. $\frac{\partial}{\partial t}g = 0$;

iii) The mean curvature vanishes for all u, v, and t.

Exercise 2. (7 points)
For $t > 0$ consider the *tractrix* $(r, h)(t) := \left(\frac{1}{\cosh(t)}, t - \tanh(t)\right)$ and the corresponding surface of rotation[1]

$$f : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}^3, (t, \varphi) \mapsto (r(t)\cos(\varphi), r(t)\sin(\varphi), h(t)).$$

i) Sketch the tractrix.

ii) Determine both principal curvatures κ_1 and κ_2.

iii) Show that its Gaussian curvature is constant.

iv) Determine its surface area.

Exercise 3. (4 points)
Let $f : \Omega \to \mathbb{R}^3$ be a parametrized surface with metric g, second fundamental form b and shape operator S. In each point $u \in \Omega$, the *third fundamental form* is a symmetric bilinear form h given by

$$h(v, w) := g(Sv, Sw) \text{ for all } v, w \in T_u\Omega.$$

Show the following equality

$$h(v, w) - 2Hb(v, w) + Kg(v, w) = 0,$$

where K denotes the Gaussian and H the mean curvature of $f(\Omega)$.

Exercise 4. (4 points)
Compute the Gaussian and mean curvature for

i) the *sphere*: $(u, v) \mapsto (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$ and

ii) the *torus*: $(u, v) \mapsto ((R + r\cos(u))\cos(v), (R + r\cos(u))\sin(v), r\sin(u))$ for constants $0 < r < R$ (cf. sheet 3).

[1] It is also called surface of revolution and this specific example is called *tractroid* or *pseudosphere*.