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Version: 1

Differential Geometry I
Winter Semester 2023/2024
Freie Universität Berlin

Exercise Sheet 2

Submission: 07.11.2023, 12:15 PM (start of lecture)

Exercise 1. (8 points)
The map

$$\eta : \left[0, \frac{2\pi}{\alpha}\right] \to \mathbb{R}^3, t \mapsto \begin{pmatrix} r\cos(\alpha t) \\ r\sin(\alpha t) \\ ht \end{pmatrix}$$

parametrizes a so called *helix* with radius $r > 0$, $\alpha \in \mathbb{R} \setminus \{0\}$, and slope $h \in \mathbb{R}$.

i) Determine $L\left(\eta|_{[0, \frac{2\pi}{\alpha}]}\right)$.

ii) Find a parametrization by arc length of η.

iii) Let $\alpha = 1$. Determine all r and h such that η is parametrized by arc length already. Sketch your results.

iv) Determine the curvature and torsion of η.

Exercise 2. (4 points)
Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a regular curve. Show that if all tangents of γ intersect in one point then γ is a straight line.

Exercise 3. (4 points)
Let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve with Frenet frame $\{\gamma'(s), n(s), b(s)\}$. For a smooth function φ the one-parameter family of rotations

$$\begin{pmatrix} \tilde{n}(s) \\ \tilde{b}(s) \end{pmatrix} = \begin{pmatrix} \cos\varphi(s) & -\sin\varphi(s) \\ \sin\varphi(s) & \cos\varphi(s) \end{pmatrix} \begin{pmatrix} n(s) \\ b(s) \end{pmatrix}$$

generates a new orthonormal frame $\left\{\gamma(s), \tilde{n}(s), \tilde{b}(s)\right\}$.

i) Compute the torsion $\tilde{\tau}(s) = \left\langle \tilde{n}'(s), \tilde{b}(s) \right\rangle$ of the new frame.

ii) Find a function φ such that the new frame is torsion free, i.e. $\tilde{\tau}(s) = 0$ for all s. Such a frame is called a *parallel frame*.

iii) Find a parallel frame for the arc length parametrized helix from exercise 1.