Freie Universität Berlin SoSe 22
Institut für Mathematik
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Differential Geometry II – Homework 11

Submission: July 15, 2022, 10:15 am

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1. Exercise (4 points)
Show the following statements about symmetric spaces:

1.) The fundamental group of a symmetric space is abelian.

2.) Let M be a Riemannian submanifold of a symmetric space $\overline{M}$. If M is connected, closed in $\overline{M}$, and totally geodesic, then M is symmetric.

2. Exercise (8 points)
Let $\mathcal{Q} := \{\xi = a + ib + jc + kd \mid a, b, c, d \in \mathbb{R}\}$ the set of *quaternions* where

$$i^2 = j^2 = k^2 = -1,\ ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik.$$

$\mathbb{R}^4$ equipped with this multiplication is called *Hamiltonian quaternions.* The imaginary part of $\xi = a + ib + cj + dk$ is equal to $bi + cj + dk$ and the conjugate element of ξ is $\bar{\xi} = a - bi - cj - dk$.

1.) Show that $\xi\bar{\xi} = |\xi|^2$ for $\xi \in \mathcal{Q}$ and the Euclidean norm.

2.) Conclude that $\mathbb{S}^3 = \{z \in \mathbb{Q} | z\overline{z} = 1\}$ equipped with the multiplication mentioned above is a Lie group.

3.) Consider $Q_8 := \{\xi \in \mathcal{L} \mid \xi| = 1\} \subseteq \mathcal{L} := \{\xi \in \mathcal{Q} \mid a, b, c, d \in \mathbb{Z}\}$. Describe the object given by the points in Q_8. Can you give it a suitable name?

3. Exercise (4 points)
Show that a homogeneous manifold is geodesically complete.

Total: 16