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SoSe 22

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Differential Geometry II – Homework 05

Submission: June 03, 2022, 10:15 am

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1. Exercise (4 points)
Let $\mathcal{X}(M)$ denote the set of all differetiable vector fields on a manifold M. Show: An $\mathbb{R}$-linear map A from $\mathcal{X}(M)^s$, $s \in \mathbb{N}$, to the set of scalar functions on M is a $(0, s)$ tensor if and only if

$$A(f_1 \cdot X_1, \ldots, f_s \cdot X_s)|_p = f_1(p) \cdot \ldots \cdot f_s(p) \cdot A(X_1, \ldots, X_s)|_p$$

holds for arbitrary scalar fundtions $f_1, \ldots, f_s$ on M in every $p \in M$.

2. Exercise (4 points)
Let X, U, V, W be vector fields on a Riemannian manifold. Show that the *Riemannian curvature tensor* R fulfils the following properties:

1.) $(\nabla_U R)(V, W)X + (\nabla_V R)(W, U)X + (\nabla_W R)(U, V)X = 0$ (second Bianchi identity),

2.) $g(R(U, V)W, X) = -g(R(U, V)X, W)$,

3.) $g(R(U, V)W, X) = g(R(W, X)U, V)$.

3. Exercise (3 points)
Let ∇ and $\tilde{\nabla}$ be two connections on a manifold M. Show that $\nabla - \tilde{\nabla}$ is a (1,2)-tensor field on M. Show that the assignment $(X, Y) \mapsto \nabla_X Y$ is not a $(1, 2)$-tensor for vector fields X, Y.

Please, turn over.

4. Exercise (5 points)

For $k \in \{-1, 0, 1\}$, consider the Riemannian manifolds $(\mathbb{S}^n \setminus \{(0, \ldots, 0, 1)\}, g_1)$, $(\mathbb{R}^n, g_0)$, and $(\mathbb{H}^n, g_{-1})$ where

$$g_k|_p := \frac{4}{(1 + k \cdot |p|^2)^2} \delta_{ij}.$$

Show: The curvature tensors R_k of the Riemannian manifolds $(\mathbb{S}^n \setminus \{(0, \ldots, 0, 1)\}, g_1)$, $(\mathbb{R}^n, g_0)$, and $(\mathbb{H}^n, g_{-1})$ satisfy

$$g_k (R_k(X, Y)Y, X) = k \cdot A^2(X, Y)$$

for all $X, Y \in T_p\mathbb{S}^n$, $T_p\mathbb{R}^n$, or $T_p\mathbb{H}^n$, respectively, where

$$A^2(X, Y) = g_k(X, X)g_k(Y, Y) - g_k(X, Y)^2.$$

Total: 16