Freie Universität Berlin Summer term 2022
Institut für Mathematik
Prof. Dr. K. Polthier, H.-S. Lipschütz Status: May 17, 2022

Differential Geometry II – Homework 04

Submission: May 25, 2022, 10:15 am

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1. Exercise (5 points)

Let (M, g) be a Riemannian manifold, $p \in M$ and $E_1, \ldots, E_n$ be an orthonormal basis of T_pM. This basis induces an isomorphism $E : \mathbb{R}^n \to T_pM$, $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i E_i$. On a (sufficiently) small neighbourhood U of p where $\exp_p$ is bijective, $x := E^{-1} \circ \exp_p^{-1} : U \to \mathbb{R}^n$ is called a *normal coordinate system* for M at p. Show:

1.) The coordinates of p are $(0, \ldots, 0)$, and $g = (\delta_{ij})$ at p.

2.) For any $V = \sum_{i=1}^n V_i \partial_i \in T_pM$, the geodesic γ emanating from p with initial velocity V is given by $x(\gamma(t)) = (tV_1, \ldots, tV_n)$.

3.) The first partial derivatives of g_{ij} and the Christoffel symbols vanish at p.

2. Exercise (4 points)

Let (M, g) be a Riemannian manifold with Levi Civitŕ connection ∇. Let R be the $(1, 3)$-curvature tensor

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

1.) Show that R is tensorial in Z, i.e. $R(X, Y)(fZ) = f\, R(X, Y)Z$ for a C^∞ function f.

2.) Determine the coordinates ${R_{ijk}}^\ell$ of R in the coordinate expression $R(\partial_i, \partial_j)\partial_k = \sum_\ell {R_{ijk}}^\ell\, \partial_\ell$.

3. Exercise (2 points)

Let X be a fixed vector field. Show that the covariant derivative $\nabla_X R$ of the curvature tensor is again a $(1, 3)$-tensor given by

$$(\nabla_X R)(U, V)W = \nabla_X(R(U, V)W) - R(\nabla_X U, V)W - R(U, \nabla_X V)W - R(U, V)\nabla_X W.$$

Please turn over.

4. Exercise (5 points)

Consider the open unit disk in $\mathbb{R}^2$ given in polar coordinates $\{(r, \varphi) \in [0, 1[\times [0, 2\pi[\}$ with the following metric

$$g = \frac{4}{(1-r^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

1.) Sketch ∂_r and ∂_φ and determine $|\partial_r|$ and $|\partial_\varphi|$.

2.) Determine $\nabla_{\partial_r}\partial_r$, $\nabla_{\partial_r}\partial_\varphi$, $\nabla_{\partial_\varphi}\partial_r$, $\nabla_{\partial_\varphi}\partial_\varphi$, and $\nabla_V W$ for $V = r\partial_r + r^2\partial_\varphi$ and $W = \varphi\partial_r + r\varphi\partial_\varphi$. Why do two of these derivatives coincide?

Total: 16