Freie Universität Berlin
Institut für Mathematik
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Differential Geometry II – Homework 03

Submission: May 18, 2022, 10:15 am

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1. Exercise (4 points)

Show the following statements:

1.) For two derivations X and Y and a smooth function we define $XY(f) := X(Y(f))$. Find two derivations X and Y on $\mathbb{R}^2$ such that XY is *not* a derivation.

2.) Show that the Lie bracket $[X, Y]$ of two vector fields is again a derivation at each point.

2. Exercise (6 points)

Let $U, V, W \in \mathrm{TM}$ be vectorfields on a k-manifold M and let $f, h : M \to \mathbb{R}$ be differentiable functions. Show the following properties of the Lie bracket:

1.) $[U, V] = -[V, U]$,

2.) $[fU, hV] = f \cdot h \cdot [U, V] + f \cdot U(h) \cdot V - h \cdot V(f) \cdot U$,

3.) $[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0$,

4.) $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ for any chart with coordinates $(x^1, \ldots, x^k)$,

5.) coordinate representation: $[V, W] = \sum_{i,j=1}^{k}(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i})\frac{\partial}{\partial x^j}$.

3. Exercise (6 points)

Consider $\mathbb{R}^n$ with the standard metric. Show that the directional derivative D given at $p \in \mathbb{R}^n$ by

$$D_V W|_p := \lim_{t \to 0} \frac{W|_{p+tV} - W|_p}{t} = DW \cdot V|_p$$

is a Riemannian connection.

Total: 16