Freie Universität Berlin Summer term 2022
Institut für Mathematik
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Differential Geometry II – Homework 02

Submission: May 11, 2022, 10:15 am

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1. Exercise (4 points)
Let $\mathcal{A}$ be a smooth atlas for a manifold M and let $\overline{\mathcal{A}}$ be the set of all charts that are smoothly compatible[1] with every chart in $\mathcal{A}$. Show that $\overline{\mathcal{A}}$ is the unique maximal smooth atlas containing $\mathcal{A}$.

2. Exercise (4 points)
Find two smooth atlases for the real line $\mathbb{R}$ which are not smoothly compatible to each other and thus yield two different smooth structures for $\mathbb{R}$. Justify your solution.

3. Exercise (5 points)
Let $p \in \mathbb{R}^k$. A *derivation* at p is a map $X|_p : C^\infty(\mathbb{R}^k) \to \mathbb{R}$ that satisfies the following conditions:

* $X|_p(\alpha f + \beta g) = \alpha X|_p(f) + \beta X|_p(g)$ for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty\left(\mathbb{R}^k\right)$ (linearity),
* $X|_p(f \cdot g) = X|_p(f) \cdot g + f \cdot X|_p(g)$ for all $f, g \in C^\infty(\mathbb{R}^k)$ (Leibniz rule).

1.) Show $X|_p(f) = 0$ for f constant.

2.) Show $X|_p(f \cdot g) = 0$ for $f(p) = g(p) = 0$.

3.) Show that the set $T_p(\mathbb{R}^k)$ of all derivations at p is a k-dimensional real vector space. Therefore, consider the map sending $v \in \mathbb{R}^k$ to the derivative in direction of v.

Please turn over.

[1] A chart (U, φ) is *smoothly compatible* with a chart (V, ψ) if either $U \cap V = \varnothing$ or the change of coordinates $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

4. Exercise (3 points)

Consider the following parameterization of the *hyperbolic paraboloid*

$$h : \mathbb{R}^2 \to \mathbb{R}^3,\ (u, v) \mapsto \begin{pmatrix} u \\ v \\ u^2 - v^2 \end{pmatrix}.$$

Further, denote with u^i, $i \in \{1, 2\}$, the standard coordinates in $\mathbb{R}^2$.

1.) Use the given parameterization to derive a (single) chart φ to represent the image of $\mathbb{R}^2$ under f as a differnetiable 2-manifold M.

2.) Derive coordinates (x^1, x^2) for M explictly, where $x^i(p) = u^i(\varphi(p))$ for $p \in M$ and $i \in \{1, 2\}$.

3.) Determine the derivatives $\frac{\partial f}{\partial x^i}$, $i \in \{1, 2\}$, at a point $p \in M$ for the function $f : M \to \mathbb{R}$ which is the restriction to M of the function $(x, y, z) \mapsto z$ on $\mathbb{R}^3$ explicitly.

Total: 16