Freie Universität Berlin
Institut für Mathematik
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Differential Geometry II – Homework 01

Submission: May 04, 2022, 10:15 am

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1. Exercise (7 points)

Consider the *real projective plane* $\mathbb{R}P^2 = \mathbb{S}^2/\sim$ where

$$p \sim q \;\Leftrightarrow\; x = y \text{ or } x = -y \text{ for all } x, y \in \mathbb{S}^2.$$

1.) Show that $\sim$ is an equivalence relation.

2.) Construct an atlas for $\mathbb{R}P^2$ consisting of exactly three charts explicitly.

2. Exercise (9 points)

Consider the *special orthogonal group* SO(3) consisting of all orthogonal matrices whose determinant is equal to 1. Show the following parts to show that SO(3) is a 3-manifold:

1.) Consider the *Cayley map* given by

$$\mathrm{Cay} : \mathbb{R}^3 \to \mathrm{SO}(3), \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto (I_3 + A) \cdot (I_3 - A)^{-1},$$

where $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ is skew symmetric and I_3 denotes the 3×3 identity matrix.

a) Show that $\mathrm{Cay}(A) \in \mathrm{SO}(3)$ for arbitrary $(a, b, c)^T \in \mathbb{R}^3$.

b) Show that Cay is injective.

2.) Characterize all elements of SO(3) not lying in the image of Cay.

3.) Use the first exercise to cover these elements.

4.) Determine the number of necessary charts.

5.) Show that the constructed charts form an atlas of SO(3).

Total: 16