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Complex Analysis – Homework 12

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1. Exercise (6 points)
Consider the following pairs of sets P_k and Q_k, $k \in \{1, 2, 3, 4\}$, and decide whether there exists a biholomorphic map $\varphi_k : P_k \to Q_k$. In either case, justify your solution:

1.) $P_1 := \{z \in \mathbb{C} \mid |z| < 1\}$ and $Q_1 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$,

2.) $P_2 := \mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0\}$ and $Q_2 := \mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathrm{Im}(z) = 0\}$,

3.) $P_3 := \{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < 1\}$ and $Q_3 := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$,

4.) $P_4 := \{z \in \mathbb{C} \mid |z| < 1\}$ and $Q_4 := \mathbb{C}$.

2. Exercise (4 points)
Let $f : \mathbb{C} \to \mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathrm{Im}(z) = 0 \text{ and } \mathrm{Re}(z) \geq 0\}$ be a holomorphic function. Determine $f(\mathbb{C})$.

3. Exercise (6 points)
Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$.

1.) Let $f : \mathbb{D} \to \mathbb{D}$ be a biholomorphic map. Show that in this case, f is of the following form:

$$f(z) = \exp(i\varphi)\frac{z - z_0}{1 - \overline{z_0}z},$$

where $\varphi \in \mathbb{R}$ and $z_0 \in \mathbb{D}$.

2.) Let $R > 0$ and $U := \left\{z = r\exp(i\varphi) \in \mathbb{C} \mid r \in (0, R) \text{ and } \varphi \in \left(0, \frac{\pi}{10}\right)\right\}$. Construct a biholomorphic map $f : U \to \mathbb{D}$ explicitly.

Total: 16