**Freie Universität Berlin** SoSe 21
**Institut für Mathematik**
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# Complex Analysis – Homework 11

Submission: July, 6th, 2021, 10:15 am, via email

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**1. Exercise** (4 points)
Show the following theorem stated in the lecture: The meromorphic functions defined on $\hat{\mathbb{C}}$ are exactly the rational functions.

**2. Exercise** (4 points)
For $k \in \{1, 2, 3\}$, let $\Omega_k \subset \mathbb{C}$ be maximal domains such that the following functions $f_k : \Omega_k \to \hat{\mathbb{C}}$ are holomorphic in $\Omega_k$ except for a discrete set of isolated singularities. Determine

1.) $\mathrm{res}_{z_0} f_1$ where $f_1(z) := z^k$, $k \in \mathbb{Z}$, and $z_0 = 0$,

2.) $\mathrm{res}_{z_0} f_2$, where $f_2(z) := \frac{5z-1}{z(z^2-1)}$ for all singularities $z_0$ of $f_2$,

3.) $\int_{\gamma(0,1/2)} f_3(z)\, dz$, where $f_3(z) := \frac{3z^2-1}{z^3-z}$ using the residue theorem.

**3. Exercise** (4 points)
Let $B_r^\circ(a) := B_r(a) \setminus \{a\}$ denote the punctured neighborhood of radius $r \in \mathbb{R}_{>0}$ of some singularity $a \in \mathbb{C}$. Show the following statements:

1.) The residue $\mathrm{res}_a f$ is the uniquely defined number $c \in \mathbb{C}$, such that the function

$$g(z) := f(z) - \frac{c}{z-a}$$

has an antiderivative in some punctured neighborhood of $a$.

2.) Let $f$ be a holomorphic function on $B_r^\circ(0)$. Then $\mathrm{res}_0 f' = 0$.

*Please, turn over.*

**4. Exercise** (4 points)

Let $\Omega \subseteq \mathbb{C}$ be a domain and $\gamma : [0,1] \to \Omega$ a smooth closed curve. Show that for $a \notin \mathrm{tr}(\gamma)$, the winding number $n(\gamma, a)$ is an integer, i.e.

$$n(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - a}\, d\xi \in \mathbb{Z}.$$

*Hint:* For $t \in [0,1]$, consider the functions

$$G(t) := \int_0^t \frac{\gamma'(s)}{\gamma(s) - a}\, ds \text{ and } F(t) := (\gamma(t) - a)\exp(-G(t)).$$

Total: 16