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Complex Analysis – Homework 7

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1. Exercise (4 points)

Determine the maxima $|f_k|$, $k \in \{1, 2, 3, 4\}$, on the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} \mid |z| \leq 1\}$ of the following functions:

1.) $f_1(z) = \exp(z^2)$,

2.) $f_2(z) = \frac{z+3}{z-3}$,

3.) $f_3(z) = z^2 + z - 1$,

4.) $f_4(z) = 3 - |z|^2$. Does f_4 contradict the maximum principle?

2. Exercise (4 points)

Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the (open) unit disk. Further, let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Show that $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$.

3. Exercise (8 points)

As presented in the lecture, the trigonometric functions $\sin, \cos : \mathbb{C} \to \mathbb{C}$ are defined via the following power series:

$$\sin(z) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \text{ and } \cos(z) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}.$$

Show:

1.) Both series converge for all $z \in \mathbb{C}$.

2.) They are the only holomorphic extensions of the real functions $\sin_{\mathbb{R}}, \cos_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$.

3.) The well-known derivation rules $\sin' = \cos$ and $\cos' = -\sin$ hold in the complex case as well.

4.) Determine the integral

$$\int_{\gamma_{(0,3)}} \frac{\cos(\pi z)}{z^2 - 1},$$

where $\gamma_{(0,3)}$ denotes the circle with center 0 and radius 3.

Total: 16