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Complex Analysis – Homework 6

Submission: June, 1st, 2021, 10:15 am, via email

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1. Exercise (4 points)
Let $\Omega \subseteq \mathbb{C}$ be a domain, $I := [a, b] \subset \mathbb{R}$ an interval, and $f : I \times \Omega \to \mathbb{C}$ a continuous function such that for fixed $t \in I$ the parameterized function $f(t, .) : \Omega \to \mathbb{C}$ is holomorphic. Further, the partial derivative $\frac{\partial f}{\partial z} : I \times \Omega \to \mathbb{C}$ is continuous.
Show the following statements about the parametric integral $g(z) := \int_a^b f(t, z)\, dt$:

1.) g is holomorphic on Ω.

2.) Differentiation and integration can be exchanged, i.e. it holds that $g'(z) = \int_a^b \frac{\partial f(t,z)}{\partial z}\, dt$.

2. Exercise (4 points)
Let $f, g : \mathbb{C} \to \mathbb{C}$ be entire functions and $f(g(z)) \equiv 0$ for all $z \in \mathbb{C}$.

1.) Show that $f \equiv 0$ if g is not constant.

2.) Find a real counterexample–i.e. two functions $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ with continuous partial differentiable component functions where g is not constant, but $f(g(x, y)) = 0$ for all $(x, y) \in \mathbb{R}^2$.

3. Exercise (4 points)
Let $\Omega \subset \mathbb{C}$ be a domain which is closed under complex conjugation (i.e. $z \in \Omega \Rightarrow \overline{z} \in \Omega$). Define the following subset of Ω: $\Omega^+ := \{z \in \Omega \mid \mathrm{Im}(z) > 0\}$, $\Omega^- := \{z \in \Omega \mid \mathrm{Im}(z) < 0\}$, and $\Omega^0 := \{z \in \Omega \mid \mathrm{Im}(z) = 0\}$. Further, let $f : \Omega^+ \cup \Omega^0 \to \mathbb{C}$ be a continuous function which is holomorphic on Ω^+ and fulfills $f(\Omega^0) \subseteq \mathbb{R}$.
Show that the extension

$$\tilde{f}(z) := \begin{cases} f(z), & \text{if } z \in \Omega^+ \cup \Omega^0, \\ \overline{f(\overline{z})}, & \text{if } z \in \Omega^-, \end{cases}$$

is a holomorphic function on whole Ω.

Please, turn over.

4. Exercise (4 points)

1.) Show that every rational function $f(z) = \frac{p(z)}{q(z)}$ where p, q are polynomials and $q \not\equiv 0$ can be written as a sum of a polynomial and a finite linear combination rational functions of the following form

$$z \mapsto \frac{1}{(z-a)^k} \text{ for } a \in \mathbb{C},\ k \in \mathbb{N}.$$

2.) Derive such a decomposition for $f(z) = \frac{2\left(z^2-z\right)+1}{z^2-2z+1}$ explicitly.

Total: 16