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Complex Analysis – Homework 5

Submission: May, 25th, 2021, 10:15 am, via email

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1. Exercise (4 points)

Let $\gamma_{(m,r)} : [0, 2\pi] \to \mathbb{C}$, $t \mapsto m + r\exp(it)$, be the curve parameterizing the circle with center $m \in \mathbb{C}$ and radius $r \in \mathbb{R}_{>0}$. Calculate the following integrals by using Cauchy's integral theorem and integral formula:

1.) $\int_{\gamma_{(2,1)}} \frac{z^7+1}{z^2(z^4+1)}\,dz$,

2.) $\int_{\gamma_{(i,1)}} \frac{\exp(z)}{z^2+1}\,dz$,

3.) $\int_{\gamma_{(1,1)}} \left(\frac{z}{z-1}\right)^n dz$, $n \in \mathbb{N}$,

4.) $\int_{\gamma_{\left(0,\frac{1}{2}\right)}} \frac{\exp(1-z)}{z^3(1-z)}\,dz$.

2. Exercise (4 points)

1.) Decide which of the following sets are star domains. If a set is a star domain determine the set of points which can be connected to all points in the set by a straight line:

   a) $S_1 = \left\{z \in \mathbb{C} \mid |z| < 1 \text{ and } |z+1| > \sqrt{2}\right\}$,

   b) $S_2 = \left\{z \in \mathbb{C} \mid |z| < 1 \text{ and } |z-2| > \sqrt{5}\right\}$,

   c) $S_3 = \{z \in \mathbb{C} \mid |z| < 2 \text{ and } |z+i| > 2\}$.

2.) Let $S, \tilde{S} \subseteq \mathbb{C}$ be star domains and $z^* \in \mathbb{C}$ be a point such that all points in S and $\tilde{S}$ can be connected to with a straight line. Decide whether $S \cap \tilde{S}$ and $S \cup \tilde{S}$ are star domains (w.r.t. z^*).

Please turn over.

3. Exercise (4 points)

Let $S \subseteq \mathbb{C}$ be a star domain and let $z^* \in S$ such that all points in S can be connected to z^* with a straight line. Show:

1.) Let $a \in S$. Then there exists a neighborhood U_a of a which is completely contained in S such that all for $z \in U_a$ the three points z^*, z, and a form a (possibly degenerate) triangle which is contained completely in U_a.

2.) Show that Cauchy's integral theorem holds for star domains.

4. Exercise (4 points)

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $G \subset \mathbb{C} \setminus \{0\}$ be a domain containing $\bar{\mathbb{D}} \setminus \{0\}$. Further, let $f : G \to \mathbb{C}$ be a holomorphic map that is bounded on $\bar{\mathbb{D}} \setminus \{0\}$. Show:

$$\int_{\partial \mathbb{D}} f(z)\, dz = 0.$$

Total: 16