Freie Universität Berlin SoSe 21
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Complex Analysis – Homework 3

Submission: May, 11th, 2021, 10:15 am, via email

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1. Exercise (5 points)

The so-called *Cayley transform* is a Möbius transformation given by

$$f_C : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad z \mapsto \frac{z-i}{z+i}.$$

1.) Sketch the images under f_C of those lines which are parallel to the real or imaginary axis.

2.) Determine f_C^{-1}.

3.) Conclude that f_C is a biholomorphic map (i.e. bijective holomorphic) from the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ to the unit disk $\mathbb{D} := \{w \in \mathbb{C} \mid |w| < 1\}$.

2. Exercise (4 points)

Let $p(z)$ be a complex polynomial and $z_0 \in \mathbb{C}$ arbitrary, but fixed, and $r > 0$. Show:

$$\int_\gamma p(z)\,dz = 0 \quad \text{and} \quad \int_\gamma \overline{p(z)}\,dz = 2\pi i r^2\,\overline{p'(z_0)},$$

where $\gamma(t) = z_0 + r\exp(it)$, $t \in [0, 2\pi]$.

3. Exercise (3 points)

Investigate whether $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}^2$, has an antiderivative.

4. Exercise (4 points)

Let $G_1, G_2 \subset \mathbb{C}$ be two regions and $f : G_1 \cup G_2 \to \mathbb{C}$ be a continuous function. For every closed path in G_1 or G_2, resp., $\int_\gamma f(z)\,dz = 0$ holds.

Show: If $G_1 \cap G_2$ is connected, then $\int_\gamma f(z)\,dz = 0$ for every closed path γ in $G_1 \cup G_2$.

Total: 16