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Complex Analysis – Homework 2

Submission: May, 4th, 2021, 10:15 am, via email

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1. Exercise (7 points)

1.) Determine the Möbius transformation f explicitly which fulfils $f(0) = i$, $f(i) = \infty$, and $f(\infty) = 1$. Sketch the images of the following subsets of $\hat{\mathbb{C}}$ under f:

a) $\mathcal{Q}_1 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > 0\}$,

b) $\mathcal{Q}_2 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) < 0\}$,

c) $\mathcal{Q}_3 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0,\ \mathrm{Im}(z) > 0\}$,

d) $\mathcal{Q}_4 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0,\ \mathrm{Im}(z) < 0\}$.

Where are the real and the imaginary axis mapped to?

2. Exercise (5 points)

1.) Show that every Möbius transformation has at least one fixed point.

2.) Let f be a Möbius transformation with exactly one fixed point. Show that there exists a Möbius transformation g and $\alpha \in \mathbb{C}$ such that

$$\left(g^{-1} \circ f \circ g\right)(z) = z + \alpha.$$

3. Exercise (4 points)

Consider the following elementary transformations $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$$\tau_\lambda(z) := z + \lambda, \quad \rho(z) := \tfrac{1}{z}, \quad \sigma_\lambda(z) := \lambda z \text{ for } \lambda \in \mathbb{C} \text{ fixed.}$$

1.) What do τ_λ, ρ, and σ_λ geometrically cause? Give a short description.

2.) Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation with $c \neq 0$. Show that

$$f(z) = (\tau_{\lambda_1} \circ \sigma_{\lambda_2} \circ \rho \circ \tau_{\lambda_3})(z) \text{ where } \lambda_1 = \tfrac{a}{c},\ \lambda_2 = \tfrac{bc-ad}{c^2},\ \lambda_3 = \tfrac{d}{c}.$$

Total: 16