Freie Universität Berlin WiSe 2021/22
Institut für Mathematik
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Differential Geometry I – Homework 02

Submission: November 11, 2021, 12:00 noon

1. Exercise (4 points)

Let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve with Frenet frame $\{T(s), N(s), B(s)\}$. For a smooth function ϕ the one parameter family of rotations

$$\begin{pmatrix} \tilde{N}(s) \\ \tilde{B}(s) \end{pmatrix} = \begin{pmatrix} \cos(\phi(s)) & -\sin(\phi(s)) \\ \sin(\phi(s)) & \cos(\phi(s)) \end{pmatrix} \begin{pmatrix} N(s) \\ B(s) \end{pmatrix}$$

generates a new orthonormal frame $\{T(s), \tilde{N}(s), \tilde{B}(s)\}$.

1.) Compute the torsion $\tilde{\tau}(s) := \langle \tilde{N}'(s), \tilde{B}(s) \rangle$ of the new frame $\{T(s), \tilde{N}(s), \tilde{B}(s)\}$.

2.) Determine some function ϕ such that the frame $\{T(s), \tilde{N}(s), \tilde{B}(s)\}$ is torsion-free.

2. Exercise (4 points)

Let $\gamma : I \to \mathbb{R}^3$ be a curve with $\gamma'' \neq 0$ everywhere and κ, τ the curvature and torsion defined via its Frenet frame. Suppose $\tau \neq 0$ everywhere. Show that if

$$\frac{\tau}{\kappa} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\kappa'}{\tau \kappa^2} \right),$$

the curve γ lies on a sphere, i.e. $\gamma \cdot \gamma = r^2$, where $r \in \mathbb{R}_+$.

3. Exercise (4 points)

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a regular curve. Show: for all $\epsilon > 0$, there exists $\delta > 0$ such that for any subdivision $a \leq t_0 < t_1 < \cdots < t_n \leq b$ with resolution $\max_i |t_{i+1} - t_i| < \delta$ the maximum distance between curve and polygon $P = (\gamma(t_0), \ldots, \gamma(t_n))$ is smaller than ϵ:

$$\max_{t \in [a,b]} |\gamma(t) - P(t)| < \epsilon.$$

4. Exercise (4 points)

1.) For $a, b, c \in \mathbb{R}$ compute first and second fundamental forms of the following surfaces of rotation:

a) the ellipsoid:

$$f_1 : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3, (u, v) \mapsto (a \sin(u) \cos(v), b \sin(u) \sin(v), c \cos(u)).$$

b) the catenoid:

$$f_2 : \mathbb{R} \times [0, 2\pi] \to \mathbb{R}^3, (u, v) \mapsto (a \sinh(u) \cos(v), b \sinh(u) \sin(v), c \cosh(u)).$$

2.) (3 additional points) Plot the surfaces corresponding to f_1 and f_2.