Freie Universität Berlin WiSe 2021/22
Institut für Mathematik
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Differential Geometry I – Homework 01

Submission: November 4, 2021, 12:00 noon

1. Exercise (4 points)

Show the following properties of the *Bernstein polynomials*:

1.) Symmetry: $B_k^n(t) = B_{n-k}^n(1-t)$,

2.) Partition of unity: $\sum_{k=0}^n B_k^n(t) = 1$,

3.) Derivative: $\frac{\mathrm{d}}{\mathrm{d}t} B_k^n(t) = n\left(B_{k-1}^{n-1}(t) - B_k^{n-1}(t)\right)$, and

4.) Basis of polynomials: the Bernstein polynomials $\{B_0^n(t), B_1^n(t), \dots, B_n^n(t)\}$ of degree n form a basis of the vector space of polynomials of degree n. *Hint: Use c) to show the linear independence of the $n+1$ Bernstein polynomials.*

2. Exercise (6 points)

Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$ be a planar Frenet curve with Frenet frame $\{T(t), N(t)\}$. Its evolute η is defined by

$$\eta(t) := \gamma(t) + \frac{1}{\kappa(t)} N(t).$$

1.) Show that the tangent at every $t \in I$ of the evolute η is normal to Frenet curve γ at t.

2.) Compute and sketch the evolute of an ellipse: $\gamma_1 : [0, 2\pi] \to \mathbb{R}^2$ with $\gamma_1(t) = (a\cos(t), b\sin(t))$, where $a, b \in \mathbb{R}$.

3.) Let $\gamma_2 : (0, 2\pi) \to \mathbb{R}^2$ defined by $\gamma_2(t) = (t - \sin(t), 1 - \cos(t))$ be the cycloid. Compute and sketch its evolute.

3. Exercise (3 points)

Let $\gamma : [0, \infty) \to \mathbb{R}^2$ defined by $\gamma(t) := (ae^{bt}\cos(t), ae^{bt}\sin(t))$ be a parametrized curve with $a > 0$ and $b < 0$.

1.) Sketch the trace of γ.

2.) Show that γ approaches the origin for $t \to \infty$ and has finite length.

4. Exercise (3 points)

Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a regular curve. Show: if all tangents of γ intersect in one point $p \in \mathbb{R}^2$ then γ is a straight line.

Total: 16