Freie Universität Berlin WiSe 21/22
Institut für Mathematik
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Differential Geometry I – Homework 04

Submission: November 25, 2021, 12:15 pm

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1. Exercise (6 points)
Determine the Christoffel symbols for the following surfaces:

1.) $f_1 : [0, \pi) \times [0, 2\pi) \to \mathbb{R}^3$, $(u, v) \mapsto r(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v))$, $0 < r$,

2.) $f_2 : [0, 2\pi)\times[0, 2\pi) \to \mathbb{R}^3$, $(u, v) \mapsto ((R+r\cos(v))\cos(u), (R+r\cos(v))\sin(u), r\sin(v))$, $0 < r < R$.

2. Exercise (4 points)
Let f parameterize the unit sphere $\mathbb{S}$ centered at the origin given by Exercise 1) with $r = 1$. Determine an explicit parametrization of a geodesic c with initial values $p = c(0) = (0, 0, 1)^T$ and $c'(0) = (1, 0, 0)^T \in T_p f$, and show that c satisfies the differential equations of geodesics.

3. Exercise (6 points)
Let $c = f \circ \gamma$ be a curve parameterized by arc length which is contained in a surface patch $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$. The *Darboux frame* $\{t, b, n\}$ is defined by the relations

$$t(s) := c'(s),\ b(s) = n(s) \times t(s),\ \text{and } n(s) \text{ is the surface normal at } c(s).$$

The frame equations of this generalized frame are

$$\begin{pmatrix} t' \\ b' \\ n' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} t \\ b \\ n \end{pmatrix},$$

where κ_g is the *geodesic curvature*, κ_n the *normal curvature*, and τ_g the *geodesic torsion.*

1.) Show that the normal curvature κ_n satisfies $\kappa_n = b(\gamma', \gamma')$, and —in case that c is a Frenet curve—that its curvature satisfies $\kappa^2 = \kappa_g^2 + \kappa_n^2$.

2.) Show that c is a *line of curvature* (i.e. $c'(s)$ is a principal curvature direction for all s) if and only if the geodesic torsion vanishes, i.e. $\tau_g(s) = 0$ for all s.

3.) Show that if c is a Frenet curve and an *asymptotic line* (i.e. $\kappa_n(s) = 0$ for all s) then the geodesic torsion is equal to the torsion of the Frenet frame (i.e. $\tau_g(s) = \tau(s)$).

Total: 16