

## Differential Geometry II – Homework 10

Submission: July 16th, 2018, 12:15 am

---

### 1. Exercise

(8 points)

- 1.) Recall that the compactified complex plane  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  is diffeomorphic to  $\mathbb{S}^2$  via stereographic projection  $\Phi : \mathbb{S}^2 \rightarrow \hat{\mathbb{C}}$ . Show that

$$\pi : \mathbb{S}^3 \subset \mathbb{C}^2 \rightarrow \mathbb{S}^2, (z_1, z_2) \mapsto \Phi^{-1}\left(\frac{z_1}{z_2}\right)$$

is a smooth projection whose preimage  $\pi^{-1}(p)$  of each point  $p \in \mathbb{S}^2$  is a circle in  $\mathbb{S}^3$ , a Hopf fiber, where  $p = (x, y, z, t) \in \mathbb{S}^3 \subset \mathbb{R}^4 \hat{=} (z_1, z_2) = (x + iy, z + it) \in \mathbb{C}^2$ .

- 2.) Let  $E$  denote the equator of  $\mathbb{S}^2$ , i.e. the preimage of the unit circle  $\mathbb{S}^1 \subset \mathbb{C}$  under  $\Phi$ . Then  $\pi^{-1}(E)$  is a torus in  $\mathbb{S}^3$ , the so-called *Clifford torus*. Show the following statements:
- The Clifford torus is flat, i.e. its Gaussian curvature vanishes.
  - The Clifford torus is a minimal surface in  $\mathbb{S}^3$ .

*Hint:* Probably, *Hopf coordinates* on  $\mathbb{S}^3$  can be useful:

$$(\theta, u, v) \in [0, \frac{\pi}{2}] \times [0, 2\pi] \times [0, \pi] \mapsto \begin{pmatrix} \cos(\theta) \cos(u + v) \\ \cos(\theta) \sin(u + v) \\ \sin(\theta) \cos(u - v) \\ \sin(\theta) \sin(u - v) \end{pmatrix}.$$

### 2. Exercise

(4 points)

$SU(n)$  denotes the *special unitary group*, i.e.

$$SU(n) = \{A \in \mathbb{C}^{n,n} \mid A^H A = I_n \text{ and } \det(A) = 1\}.$$

Determine the Lie algebra of  $SU(n)$ .

### 3. Exercise

(4 points)

$X, Y$  be two left-invariant vector fields on a Lie group  $G$ . Show that the Lie bracket  $[X, Y]$  again generates a left-invariant vector field on the Lie group  $G$ .

Total: 16