Freie Universität Berlin SoSe 18
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Differential Geometry II – Homework 09

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1. Exercise (8 points)

Let $\triangle ABC$ be a Euclidean, spherical or hyperbolic triangle respectively. Denote with a, b and c the sides of $\triangle ABC$ and with $\alpha = \frac{\pi}{l}$, $\beta = \frac{\pi}{m}$ and $\gamma = \frac{\pi}{n}$ its angles, $l, m, n \geq 2$. Denote with $\triangle(l, m, n)$ the *triangle group* generated by the reflections σ_a, σ_b and σ_c.

1.) Show the following relations in $\triangle(l, m, n)$:
   a) $\sigma_a^2 = \sigma_b^2 = \sigma_c^2 = \mathrm{id}_X$,
   b) $(\sigma_c \circ \sigma_b)^l = \mathrm{id}_X$, $(\sigma_c \circ \sigma_a)^m = \mathrm{id}_X$ and $(\sigma_a \circ \sigma_b)^n = \mathrm{id}_X$, where $X \in \{\mathbb{R}^2, \mathbb{S}^2, \mathbb{H}^2\}$.

2.) Determine all possible tessellations (with triangles) for $\mathbb{S}^2$ and $\mathbb{H}^2$. Sketch your solutions.

2. Exercise (8 points)

Let $\mathcal{Q} := \{\xi = a + ib + jc + kd | a, b, c, d \in \mathbb{R}\}$ the set of *quaternions* where

$$i^2 = j^2 = k^2 = -1,\ ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik.$$

1.) Show that $(\mathcal{Q}, \cdot)$ is an algebra. Is it commutative?

2.) Conclude that $\mathbb{S}^3 = \{z \in \mathbb{Q} | z\overline{z} = 1\}$ equipped with the multiplication mentioned above is a group.

3.) Consider $Q_8 := \{\xi \in \mathcal{L} | \, |\xi| = 1\} \subseteq \mathcal{L} := \{\xi \in \mathcal{Q} | a, b, c, d \in \mathbb{Z}\}$. Describe the object given by the points in Q_8. How is it called?

Total: 16