Freie Universität Berlin SoSe 18
Institut für Mathematik
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Differential Geometry II – Homework 05

Submission: June 6, 2018, 12:15 am

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1. Exercise (4 points)

Consider the upper half plane $\{(x,y) \in \mathbb{R}\times]0,\infty[\}$ equipped with the metric

$$g = \frac{1}{y^2} g_{\text{Euclidean}} = \frac{1}{y^2}(\delta_{ij}).$$

Let γ be a parametrisation of constant speed of $\{(x_0,y)|y \in]0,\infty[\}$ for fixed x_0.

1.) Show that $J = \partial_x$ is a Jacobi field along γ.

2.) Knowing the geodesics in the upper half plane, find a non-tangential Jacobi field. Justify your solution.

2. Exercise (6 points)

Let (M,g) be a Riemannian manifold, $p \in M$ and $E_1,\dots,E_n$ be an orthonormal basis of T_pM. This basis induces an isomorphism $E : \mathbb{R}^n \to T_pM$, $(x_1,\dots,x_n) \mapsto \sum_{i=1}^n x_i E_i$. On a (sufficiently) small neighbourhood U of p where $\exp_p$ is bijective, $x := E^{-1} \circ \exp_p^{-1} : U \to \mathbb{R}^n$ is called a *normal coordinate system* for M at p. Show:

1.) The coordinates of p are $(0,\dots,0)$, and $g = (\delta_{ij})$ at p.

2.) For any $V = \sum_{i=1}^n V_i \partial_i \in T_pM$, the geodesic γ emanating from p with initial velocity V is given by $x(\gamma(t)) = (tV_1,\dots,tV_n)$.

3.) The first partial derivatives of g_{ij} and the Christoffel symbols vanish at p.

3. Exercise (4 points)

Let (M,g) be a Riemannian manifold with normal coordinates (U,x_i) around $p \in M$. For $W = \sum_{i=1}^n W_i \partial_i \in T_pM$, show that the Jacobi field along a radial geodesic γ (i.e. $\gamma(0) = p$) with $J(0) = 0$ and $J'(0) = W$ is given by $J(t) = t\sum_{i=1}^n W_i \partial_i$ for all t.

4. Exercise (2 points)

Let ∇ and $\tilde{\nabla}$ be two connections on a manifold M. Show that $\nabla - \tilde{\nabla}$ is a (1,2)-tensor field on M.

Total: 16