Freie Universität Berlin SoSe 18
Institut für Mathematik
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Differential Geometry II – Homework 02

Submission: May 7, 2018, 12:15 am

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1. Exercise (6 points)
Let $U, V, W \in \mathrm{TM}$ be vectorfields on a k-manifold M and let $f, h : M \to \mathbb{R}$ be differentiable functions. Show the following properties of the Lie bracket:

1.) $[U, V] = -[V, U]$,

2.) $[fU, hV] = f \cdot h \cdot [U, V] + f \cdot U(h) \cdot V - h \cdot V(f) \cdot U$,

3.) $[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0$,

4.) $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ for any chart with coordinates $(x^1, \dots, x^k)$,

5.) coordinate representation: $[V, W] = \sum_{i,j=1}^{k} (v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}) \frac{\partial}{\partial x^j}$.

2. Exercise (6 points)
Consider $\mathbb{R}^n$ with the standard metric. Show that the directional derivative D given at $p \in \mathbb{R}^n$ by

$$D_V W|_p := \lim_{t \to 0} \frac{W|_{p+tV} - W|_p}{t} = DW \cdot V|_p$$

is a Riemannian connection.

3. Exercise (4 points)
Show that the *Riemannian connection* has the following coordinate representaions:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k} \quad \text{and} \quad g(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = \Gamma_{ij,k}.$$

For $U = \sum_i u^i \frac{\partial}{\partial x^i}$ and $V = \sum_j v^j \frac{\partial}{\partial x^j}$, the following holds:

$$\nabla_U V = \sum_k (\sum_i u^i \frac{\partial v^k}{\partial x^i} + \sum_{i,j} \Gamma_{ij}^k u^i v^j) \frac{\partial}{\partial x^k}.$$

Total: 16