Freie Universität Berlin WiSe 17/18
Institut für Mathematik
Prof. Dr. K. Polthier, H.-S. Lipschütz Status: December 18, 2017

Differential Geometry I – Homework 08

Submission: January 8, 2017, 12:15 am

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1. Exercise (4 points)

Formulate the initial value problem for geodesics on a torus

$$f : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}^3,\ (u, v) \mapsto (\cos(u)(R + r\cos(v)), \sin(u)(R + r\cos(v)), r\sin(v)),$$

where $r, R \in \mathbb{R}_+$.

Use the differential equations to verify that the following curves on the torus are geodesics:

1.) $t \mapsto f(u_0, t)$ for u_0 constant,

2.) $t \mapsto f(t, 0)$,

3.) $t \mapsto f(t, \pi)$.

4.) Sketch the corresponding geodesics.

2. Exercise (4 points)

Let $a, b, c \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ three distinct points. Let

$$T(z) = (z, a; b, c) := \frac{z - b}{z - c} \cdot \frac{a - c}{a - b}$$

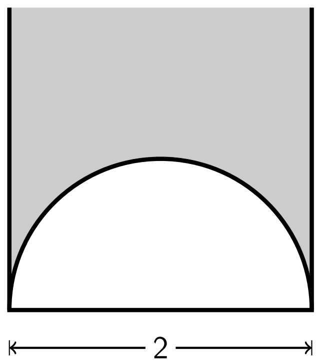
a rational linear transformation. Determine the corresponding $T(z)$ that maps i to i, ∞ to 3 and 0 to $\frac{1}{3}$.

3. Exercise (8 points)

Consider the Poincaré disk model $\mathbb{D}$.

1.) Consider the following geodesic triangle[1] in $\mathbb{D}$ whose vertices are lying all at infinity:

[1]The triangle is given by the part colored in gray.



Determine its area.

2.) Determine the area of a circle of radius $r < 1$ centered at the origin of $\mathbb{D}$.

4. Exercise (8 points)

Let $\mathbb{R}^3_L = (\mathbb{R}^3, \langle ., . \rangle_{\mathrm{L}})$ be the Lorentz space resp. the Minkowski space. Let

$$\mathbf{O}(n, 1) := \{A : \mathbb{R}^3_{\mathrm{L}} \to \mathbb{R}^3_{\mathrm{L}} \mid A \text{ preserves } \langle ., . \rangle_{\mathrm{L}}\}$$

be the so called *Lorentz group*.

1.) Show that the Lorentz group is indeed a group (resp. matrix multiplication).

2.) Show that the Lorentz group preserves $\tilde{H} := \{v \in \mathbb{R}^3_{\mathrm{L}} \mid \langle v, v \rangle_{\mathrm{L}} = -1\}$.

3.) Show that

$$\mathbf{O}_+(n, 1) := \{A \in \mathbf{O}(n, 1) \mid A \text{ preserves } \tilde{H} \cap \{v_0 > 0\}\}$$

operates on the hyperbolic space $\mathbb{H}$ and preserves its metric.

Additional exercise (4 additional points)

Let

$$f : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}^3,\ (t, \varphi) \mapsto (r(t)\cos(t), r(t)\sin(t), t)$$

parametrize a surface of revolution S and let $c : I \subsetneq \mathbb{R} \to S$ be a geodesic, $c(t) = f(r(t), \varphi(t))$. Denote with $\vartheta(t)$ the angle between $\dot{c}(t)$ and the circle of latitude running through $c(t)$. Then the following holds

$$r(t)\cos(\vartheta(t)) = \text{constant}.$$

Total: 24