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Differential Geometry II
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Freie Universität Berlin

Exercise Sheet 4

Online: 06.05.2015

Due: 13.05.2015, 4:00pm in the Tutorials

Exercise 4.1 (Tangent Bundle as a Manifold, 4 Points)**.** The *tangent bundle* TM of a differentiable manifold M is – as a set – the disjoint union of the tangent spaces of M, *i. e.*,

$$TM = \bigcup_{p \in M} \{p\} \times T_pM.$$

Prove that TM carries the structure of a differentiable manifold.

Hint: For a given chart (U, φ) on M consider the induced bundle chart given pointwise by

$$\Phi(p, X_p) := (\varphi(p), \xi_1(p), \ldots, \xi_n(p)) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where $p \in U$, $X \in T_pM$ and $\xi_1, \ldots, \xi_n$ are the coefficients of X w.r.t. the chart φ, *i. e.*, $X_p = \sum_{i=1}^n \xi_i(p) \frac{\partial}{\partial \varphi_i}|_p$ (cf. the physical interpretation of a tangent vector).

Exercise 4.2 ($T_p\mathrm{SO}(3)$, 4 Points)**.** Show that the tangent space $T_p\mathrm{SO}(3)$ of the special orthogonal group

$$SO(3) = \{A \in \mathbb{R}^{3\times 3} : \ A \cdot A^T = \mathrm{id}, \ \det A = 1\}$$

at the point $p = \mathrm{id} \in \mathrm{SO}(3)$ can be identified with the set of skew-symmetric 3×3 matrices.

Exercise 4.3 (exp I, 2 Points)**.** We consider the space $\mathbb{R}^{n\times n}$ of $n \times n$ matrices, equipped with an arbitrary matrix norm satisfying $\|AB\| \leq \|A\| \, \|B\|$ (such a function is called *sub-multiplicative*). For a matrix $A \in \mathbb{R}^{n\times n}$ we define the *matrix exponential*

$$\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Show that this series is convergent for any matrix A.

Exercise 4.4 (exp II, 2+2 Points)**.** Show that the matrix exponential maps skew-symmetric matrices to orthogonal matrices! Can you proof that it maps in fact to orthogonal matrices with determinant $+1$? (2 Bonus Points)

Combining this exercise with the previous ones this shows that exp defines a mapping $T_p\mathrm{SO}(3) \to SO(3)$.