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*Differential Geometry II*
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# Exercise Sheet 3

*Online: 29.04.2015*

*Due: 06.05.2015, 4:00pm in the Tutorials*

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**Exercise 3.1** (Not a Derivation, 2 Points)**.** For two derivations $X$ and $Y$ and a smooth function we define $XY(f) := X(Y(f))$. Find two derivations $X$ and $Y$ on $\mathbb{R}^2$ such that $XY$ is *not* a derivation.

**Exercise 3.2** (Lie Bracket, 2 Points)**.** Show that the Lie bracket $[X, Y]$ of two vector fields is again a derivation at each point.

**Exercise 3.3** (Properties of the Lie Bracket, 4 Points)**.** Prove the following properties of the Lie bracket:

1. $[fX, hY] = fh \cdot [X, Y] + f \cdot X(h) \cdot Y - h \cdot Y(f) \cdot X$;
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (*Jacobi's Identity*);
3. $\left[\sum_i \xi^i \frac{\partial}{\partial x^i}, \sum_j \eta^i \frac{\partial}{\partial x^i}\right] = \sum_{i,j} \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}$;
   (Here $\xi^i$ and $\eta^j$ are the coefficients of $X$ and $Y$ respectively in a chart $(U, x = (x^1, \ldots, x^k))$ of $M$.)
4. Conclude that $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ for any chart!

for vector fields $X, Y$ and smooth functions $f, h$ on $M$.

**Exercise 3.4** (Koszul Formula, 4 Points)**.** Derive the *Koszul formula*

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(Y, [X, Z]) - g(Z, [Y, X]) + g(X, [Z, Y])$$

for a symmetric and $g(\cdot, \cdot)$-compatible connection. Deduce that such a connection, if it exists, is uniquely determined by $g(\cdot, \cdot)$. It is called the **Levi-Cività connection** for $g(\cdot, \cdot)$.

**Hint.** Start with $X(g(Y, Z))$, $Y(g(Z, X))$, $Z(g(X, Y))$, apply $g(\cdot, \cdot)$-compatibility, then symmetry, finally sum up left- and right-hand sides.