Exercise Sheet 2

Online: 22.04.2015

Due: 29.04.2015, 4:00pm in the Tutorials

Exercise 2.1 (Unique Maximal Atlas, 4 Points)**.** Let $\mathcal{A}$ be a smooth atlas for a manifold M and let $\overline{\mathcal{A}}$ be the set of all charts that are smoothly compatible [1] with every chart in $\mathcal{A}$. Show that $\overline{\mathcal{A}}$ is the unique maximal [2] smooth atlas containing $\mathcal{A}$. Such a maximal atlas is called a *smooth structure* for the manifold M.

Exercise 2.2 (Smooth Structures on $\mathbb{R}$, 2 Points)**.** Find two smooth atlases for the real line $\mathbb{R}$ which are not smoothly compatible to each other and thus yield two different smooth structures for $\mathbb{R}$.

Exercise 2.3 (Derivations, 4 Points)**.** Let $a \in \mathbb{R}^k$. A *derivation* at a is a linear map $X : C^\infty(\mathbb{R}^k) \to \mathbb{R}$ that satisfies

$$X(fg) = f(a)\, Xg + g(a)\, Xf \quad \text{for all } f, g \in C^\infty(\mathbb{R}^k) \qquad (\textbf{Leibniz' rule}).$$

1. Show $Xf = 0$ whenever f is constant.

2. Show $X(fg) = 0$ whenever $f(a) = g(a) = 0$.

3. Show that the set $T_a(\mathbb{R}^k)$ of all derivations at a is a k-dimensional real vector space. For this purpose, consider the map sending $v \in \mathbb{R}^k$ to the derivative in direction of v.

Exercise 2.4 (Derivatives, 2 Points)**.** Consider the manifold

$$M := \{ \begin{pmatrix} u_1 \\ u_2 \\ u_1^2 - u_2^2 \end{pmatrix} : u_1, u_2 \in \mathbb{R} \} \subset \mathbb{R}^3$$

with its global chart $\varphi : M \to \mathbb{R}^2$ induced by the given parametrization. Compute the derivatives $\frac{\partial f}{\partial \varphi_i}$ at a point $p \in M$ for the function $f : M \to \mathbb{R}$ which is the restriction to M of the function $(x, y, z) \mapsto z$ on $\mathbb{R}^3$.

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[1] A chart (U, φ) is *smoothly compatible* with a chart (V, ψ) if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

[2] A smooth atlas $\mathcal{A}$ for M is *maximal* if it is not properly contained in any other smooth atlas, i.e. if every chart that is smoothly compatible with all charts in $\mathcal{A}$ is already contained in $\mathcal{A}$.