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Differential Geometry I
Winter Semester 2014/15
Freie Universität Berlin

Exercise Sheet 3

Online: 27.10.2014

Due: 5.11.2014, 4:00pm

Four points for each exercise!

Exercise 3.1 (Fundamental Forms)**.** For the following parametrized surfaces compute the first fundamental form and plot the surfaces. Furthermore compute the second fundamental form for the special case $a = b = c = 1$.

1. $f_1(u, v) := (a \sin u \cos v, b \sin u \sin v, c \cos u)$
2. $f_2(u, v) := (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$

Exercise 3.2 (Stereographic Projection)**.** Consider the sphere defined by the equation $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z - 1)^2 = 1\}$. A coordinate system is given by the *stereographic projection* $\pi : S^2 \setminus \{N\} \to \mathbb{R}^2$ which maps a point $p = (x, y, z)$ of S^2 minus the north pole $N = (0, 0, 2)$ to the intersection of the xy-plane with the straight line connecting N and p.

1. Show that $\pi^{-1} : \mathbb{R}^2 \to S^2$ is given by
$$\pi^{-1}(u, v) = \frac{2}{u^2 + v^2 + 4}(2u, 2v, u^2 + v^2).$$
2. Compute the first fundamental form of the sphere with respect to the parametrization π^{-1}.

Exercise 3.3 (Locally Conformal)**.** Let $\gamma(t) = (\rho(t), 0, z(t))$ be a regular parametrized curve in the xz-plane that does not meet the z-axis. γ defines a *surface of revolution* f by rotating γ around the z-axis:
$$f(t, \varphi) = (\rho(t) \cos \varphi, \rho(t) \sin \varphi, z(t)).$$

1. Show that a surface of revolution can always be parametrized so that $g_{11} = 1$, $g_{12} = g_{21} = 0$, $g_{22} = G(t)$ where G is a function depending on t only.
2. Show that a surface of revolution locally admits a *conformal parametrization* ($g_{11} = g_{22}$ and $g_{12} = g_{21} = 0$).

Exercise 3.4 (Surface Area)**.** Compute the surface area for the following parametrized surfaces:

1. Torus: $f(u, v) = ((R+r \cos u) \cos v, (R+r \cos u) \sin v, r \sin u)$ for constants $0 < r < R$ and $0 \leq u, v \leq 2\pi$.
2. A segment of the helicoid: $f(u, v) = (u \cos v, u \sin v, av)$ for $1 < u < 2$, $0 \leq v \leq 2\pi$ and a slant $a \in \mathbb{R}^+$.