Discrete Topology-Revealing Vector Fields on Simplicial Surfaces with Boundary

Konstantin Poelke and Konrad Polthier

Abstract We present a discrete Hodge-Morrey-Friedrichs decomposition for piecewise constant vector fields on simplicial surfaces with boundary which is structurally consistent with the smooth theory. In particular, it preserves a deep linkage between metric properties of the spaces of harmonic Dirichlet and Neumann fields and the topology of the underlying geometry, which reveals itself as a discrete de Rham theorem and a certain angle between Dirichlet and Neumann fields. We illustrate and discuss this linkage on several geometries.

1 Introduction

Hodge-type decomposition statements form an indispensable tool for the analysis and structural understanding of vector fields and more generally differential forms on manifolds. Dating back at least to Helmholtz' classical result [9] on the decomposition of a vector field into a divergence-free and a rotation-free component, there has been a remarkable evolution of extensions and generalizations. Nowadays there is a well-developed theory for Hodge decompositions of differential forms of Sobolev class (see e.g. [6] for an overview), which is of central importance e.g. for finite element Galerkin methods for problems involving vector fields such as Maxwell's equations or Navier-Stokes systems. A surprising property is the strong linkage of certain spaces of harmonic forms to the topology of the geometry, whose first encounter is given by de Rham's theorem, stating that on a closed manifold the space of harmonic k-forms is isomorphic to the k-th cohomology group with real coefficients. On a surface with non-empty boundary, the corresponding statement

Konstantin Poelke
Freie Universität Berlin e-mail: konstantin.poelke@fu-berlin.de

Konrad Polthier
Freie Universität Berlin e-mail: konrad.polthier@fu-berlin.de

applies to the spaces of harmonic Dirichlet fields $\mathscr{H}_D^1$ and Neumann fields $\mathscr{H}_N^1$, which are subspaces of all harmonic fields with certain boundary conditions imposed. However, there are now two decompositions—one including $\mathscr{H}_D^1$, the other one including $\mathscr{H}_N^1$—and in general there is no single L^2-orthogonal decomposition including both these spaces at the same time. A recent result by Shonkwiler [7, 8] identifies the reason for this non-orthogonality as the existence of non-empty subspaces representing the *interior cohomology* of the manifold (in contrast to the *cohomology induced by the boundary components*), which establishes another astonishing linkage between metric properties and the topology. In particular, the principal angles between $\mathscr{H}_N^1$ and $\mathscr{H}_D^1$ seem to act as an indicator for the influence of boundary components on the overall geometry and therefore as a theoretical shape signature.

For the numerical treatment of vector fields there is a variety of discretization strategies available, e.g. the *finite element exterior calculus* [2, 1] by Arnold et. al., which suggests a family of spaces of polynomial differential forms of arbitrary degree and generalizes classical ansatz spaces such as the Raviart-Thomas elements or Nédélec's elements, or the *discrete exterior calculus* [3] by Hirani, which defines discrete differential forms as simplicial cochains. Here we focus on a discretization by *piecewise constant vector fields* (PCVFs). Their usage and analysis in geometry processing tasks goes back at least to the work by Polthier and Preuss [5] and Wardetzky [10], and they have become a main ingredient for frame field modelling, surface parametrization or deformation modelling, just to name a few examples. A complete, structurally consistent set of Hodge-type decompositions for PCVFs on simplicial surfaces with boundary has been recently developed by Poelke and Polthier [4], and we refer the reader to this article for all details concerning discretization, implementation and numerical solving left out in section 3.

2 Hodge-type Decompositions, Topology and Duality Angles

In its modern formulation the Hodge decomposition theorem states that on a closed Riemannian manifold the space Ω^k of smooth k-forms can be decomposed L^2-orthogonally as

$$\Omega^k = \mathrm{d}\Omega^{k-1} \oplus \delta\Omega^{k+1} \oplus \mathscr{H}^k \tag{1}$$

where $\mathscr{H}^k$ is the space of harmonic k-forms satisfying $\mathrm{d}\omega = \delta\omega = 0$. Here and in the following, $\oplus$ always denotes an L^2-orthogonal direct sum. A remarkable result is de Rham's theorem which provides an isomorphism $\mathscr{H}^k \cong H^k(M)$ between the space of harmonic k-forms and the k-th cohomology group with real coefficients on M, and therefore identifies the dimension of $\mathscr{H}^k$ as a *topological* invariant.

As soon as the manifold M has a non-empty boundary $\partial M \neq \emptyset$, eq. (1) is no longer valid. Instead, the analogous splitting is now given by two decomposition statements known as the *Hodge-Morrey-Friedrichs decomposition* (see [6]) and given by

$$\begin{aligned}\Omega^k &= \mathrm{d}\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus \mathscr{H}^k \cap \mathrm{d}\Omega^{k-1} \oplus \mathscr{H}_N^k \\ &= \mathrm{d}\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus \mathscr{H}^k \cap \delta\Omega^{k+1} \oplus \mathscr{H}_D^k.\end{aligned}$$

Here, the subscript $_D$ denotes *Dirichlet boundary conditions* (i.e. the tangential part $\mathbf{t}(\omega)$ of a differential form ω has to vanish along ∂M) and $_N$ denotes *Neumann boundary conditions* (i.e. the normal part $\omega\mid_{\partial M} - \mathbf{t}(\omega)$ has to vanish along ∂M) which are imposed on the corresponding spaces. Again, there are isomorphisms $\mathscr{H}_N^k \cong H^k(M)$ and $\mathscr{H}_D^k \cong H^k(M,\partial M)$, respectively, with the latter space $H^k(M,\partial M)$ denoting the k-th *relative* cohomology of M.

With respect to the characterization of vector fields on surfaces with boundary, a natural question is whether there is a single orthogonal decomposition including $\mathscr{H}_N^1$ and $\mathscr{H}_D^1$ at the same time. To this end, we say that a surface M *is of type* $\Sigma_{g,m}$, if M is a compact orientable surface of genus $g \geq 0$ with $m \geq 1$ boundary components. We have the following result:

Lemma 1. *Let M be a surface of type $\Sigma_{0,m}$. Then there is an L^2-orthogonal decomposition*

$$\Omega^1 = \mathrm{d}\Omega_D^0 \oplus \delta\Omega_N^2 \oplus \mathrm{d}\Omega^0 \cap \delta\Omega^2 \oplus \mathscr{H}_D^1 \oplus \mathscr{H}_N^1.$$

Lemma 1 includes the common case of two-dimensional flat domains embedded in $\mathbb{R}^2$. On the other hand, if $g \geq 1$ this equation does not hold any more. A recent result by DeTurck, Gluck and Shonkwiler [7, 8] identifies subspaces of $\mathscr{H}_D^1$ and $\mathscr{H}_N^1$ representing the cohomology corresponding to the *inner topology* of M (i.e. the genus) as the defect, and this observation transfers to the discrete setting, see [4, Sec. 3.4].

3 Discrete Correspondence

Now, let M_h be a compact, orientable simplicial surface with boundary, triangulated by affine triangles, equipped with the locally Euclidean metric. Let $\mathscr{X}_h$ denote the space of PCVFs on M_h, which are given by one tangent vector X_T per affine triangle T of M_h, and let $\mathscr{L}$ and $\mathscr{F}$ denote the finite element spaces of linear Lagrange and Crouzeix-Raviart elements on M_h, respectively, with subspaces $\mathscr{L}_0 \subset \mathscr{L}$ and $\mathscr{F}_0 \subset \mathscr{F}$ consisting of all basis functions whose degrees of freedom associated to simplices on the boundary are set to zero. Then there is a discrete Hodge-Morrey-Friedrichs decomposition [4, Cor. 3.3]

$$\begin{aligned}\mathscr{X}_h &= \nabla\mathscr{L}_0 \oplus J\nabla\mathscr{F}_0 \oplus \mathscr{H}_h \cap \nabla\mathscr{L} \oplus \mathscr{H}_{h,N} \\ &= \nabla\mathscr{L}_0 \oplus J\nabla\mathscr{F}_0 \oplus \mathscr{H}_h \cap J\nabla\mathscr{F} \oplus \mathscr{H}_{h,D},\end{aligned}$$

where $\mathscr{H}_h$ is the space of all *discrete harmonic PCVFs*, defined as the L^2-orthogonal complement of $\nabla\mathscr{L}_0 \oplus J\nabla\mathscr{F}_0$ within $\mathscr{X}_h$, and J denotes a counter-clockwise (with respect to a fixed unit normal field) rotation by $\pi/2$ in the tangent plane of each triangle. There are discrete de Rham isomorphisms $\mathscr{H}_{h,N} \cong H^1(M_h)$ and $\mathscr{H}_{h,D} \cong H^1(M_h,\partial M_h)$, and an analogous statement of lemma 1 holds true for the

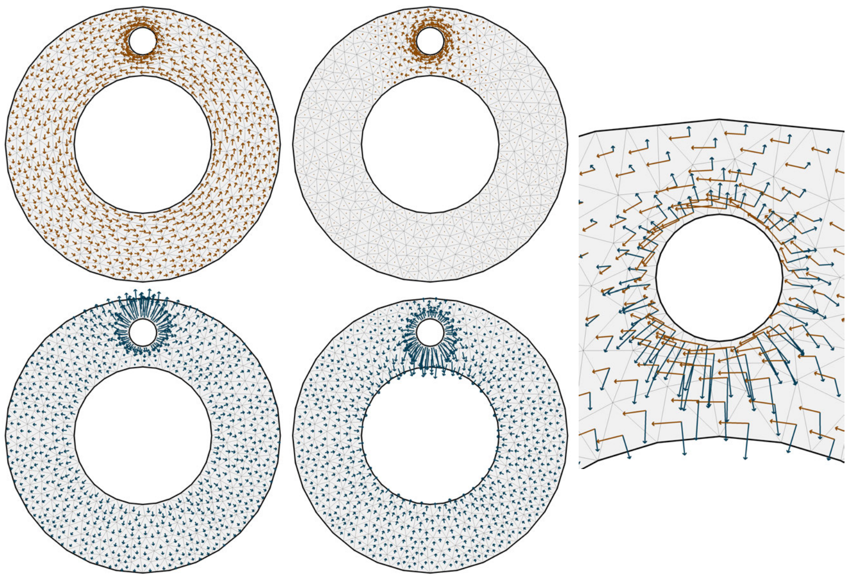

Fig. 1 Bases for $\mathscr{H}_{h,N}$ (top row) and $\mathscr{H}_{h,D}$ (bottom row) on an annulus with a hole ("AwH"), which is a surface of type $\Sigma_{0,2}$. The rightmost image shows the first Neumann field and the second Dirichlet field.

discrete version, too. In particular, by Poincaré-Lefschetz duality it is $\dim \mathscr{H}_{h,D} = \dim \mathscr{H}_{h,N} = 2g+m-1$ for a surface of type $\Sigma_{g,m}$.

Figure 1 shows bases for the spaces $\mathscr{H}_{h,D}$ and $\mathscr{H}_{h,N}$ on a surface of type $\Sigma_{0,2}$. In this case both spaces are L^2-orthogonal to each other and consequently there is a complete discrete decomposition

$$\mathscr{X}_h = \nabla \mathscr{L}_0 \oplus J\nabla \mathscr{F}_0 \oplus J\nabla \mathscr{F} \cap \nabla \mathscr{L} \oplus \mathscr{H}_{h,D} \oplus \mathscr{H}_{h,N}.$$

The numerical angles in table 1 confirm this result. Each angle α is computed as usual by

$$\cos \alpha = \frac{\langle X, Y \rangle_{L^2}}{\|X\|_{L^2} \|Y\|_{L^2}} \quad \text{for} \quad X \in \mathscr{H}_{h,N},\, Y \in \mathscr{H}_{h,D}.$$

Note that the orthogonality of the shown vector fields is always meant with respect to the L^2-product on $\mathscr{X}_h$. Locally, these fields are in general not orthogonal, see the rightmost image in fig. 1.

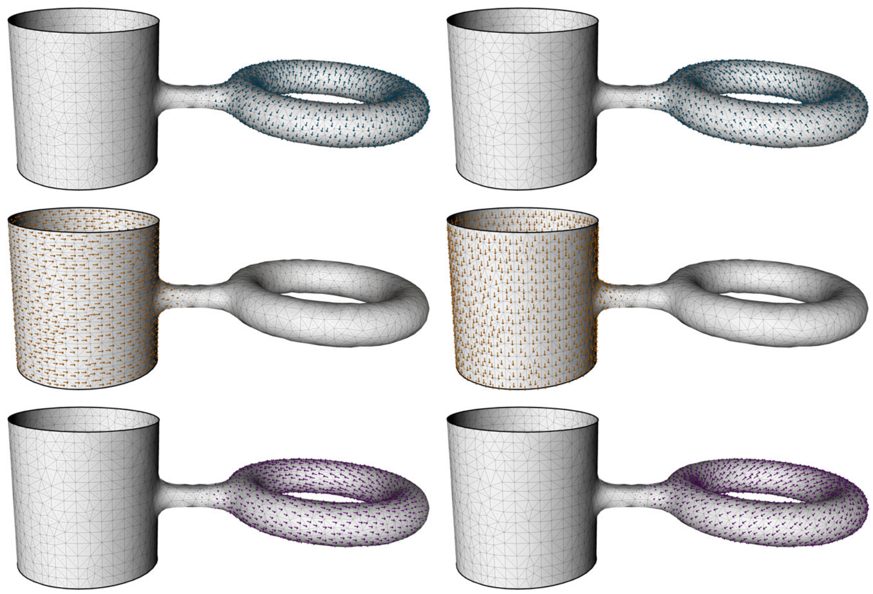

Fig. 2 Basis fields for $\mathscr{H}_{h,N}$ (left column) and $\mathscr{H}_{h,D}$ (right column) on a torus with a cylinder attached ("TwC"), which is topologically $\Sigma_{1,2}$. The fields in the first and third row all concentrate their mass in the same fashion along the longitudinal and latitudinal cycles that reflect homology generated by the torus.

Our second example in fig. 2 shows bases for the three-dimensional spaces $\mathscr{H}_{h,N}$ and $\mathscr{H}_{h,D}$ on a torus with a cylinder attached, which is of type $\Sigma_{1,2}$. Whereas both the second Neumann and Dirichlet field form an angle of almost $\pi/2$ to all other fields, this is not true for the other fields, whose masses concentrate on the toroidal region. Figure 3 shows a close-up of two pairs of fields on the toroidal region, one forming locally acute angles, the other forming locally obtuse angles. As their mass on the cylindrical region is negligible, the local situation here dominates the L^2-angle, and indeed the first pairing forms an acute L^2-angle of 0.74 radians, whereas the second pairing forms an obtuse L^2-angle of 2.31 radians, see table 1. Consequently, the spaces $\mathscr{H}_{h,N}$ and $\mathscr{H}_{h,D}$ cannot appear simultaneously in a single orthogonal decomposition on this geometry.

AwH	$\mathscr{H}_{h,D}(a)$	$\mathscr{H}_{h,D}(b)$
$\mathscr{H}_{h,N}(a)$	1.57	1.57
$\mathscr{H}_{h,N}(b)$	1.57	1.57

TwC	$\mathscr{H}_{h,D}(a)$	$\mathscr{H}_{h,D}(b)$	$\mathscr{H}_{h,D}(c)$
$\mathscr{H}_{h,N}(a)$	2.30	1.57	**0.74**
$\mathscr{H}_{h,N}(b)$	1.62	1.57	1.55
$\mathscr{H}_{h,N}(c)$	2.41	1.58	**2.31**

Table 1 Angles between the basis fields for $\mathscr{H}_{h,N}$ and $\mathscr{H}_{h,D}$ on the flat AwH-model and the TwC-model in radians.

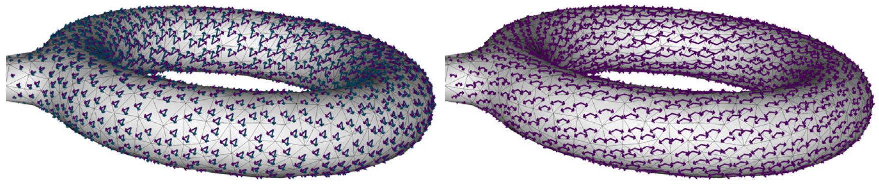

Fig. 3 Two parings of Neumann and Dirichlet fields from the bases shown in fig. 2. The left image shows the first Neumann field and the third Dirichlet field, forming locally acute angles on each triangle on the torus region. The right image shows the third Neumann field and the third Dirichlet field, forming obtuse angles.

4 Outlook

Our discretization scheme captures the structural properties present in the smooth case and in particular preserves the deep linkage between the geometry, i.e. the metric properties, and the underlying topology. Still, it is not clear in which way the angles between Dirichlet and Neumann fields are related to the boundary components of M_h. The examples explicitly computed in [7] are a first starting point for the search of a relation that could be even quantitatively described. A better understanding of this correlation is very promising with regard to applications including metric-topological shape signatures, extraction of certain vector field components with controlled characteristics and parametrization tasks of surfaces with boundary, and is current work in progress.

References

1. Douglas N. Arnold, Richard Falk, and Ragnar Winther. Finite element exterior calculus: from Hodge theory to numerical stability. *Bulletin of the American mathematical society*, 47(2):281–354, 2010.
2. Douglas N. Arnold, Richard S. Falk, and Ragnar Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, 15:1–155, 2006.
3. Anil N. Hirani. *Discrete exterior calculus*. PhD thesis, California Institute of Technology, 2003.
4. Konstantin Poelke and Konrad Polthier. Boundary-aware Hodge decompositions for piecewise constant vector fields. *Computer-Aided Design*, 78:126 – 136, 2016.
5. Konrad Polthier and Eike Preuss. Identifying vector field singularities using a discrete Hodge decomposition. In Hans-Christian Hege and Konrad Polthier, editors, *Visualization and Mathematics III*, pages 113–134. Springer Verlag, 2003.
6. Günter Schwarz. *Hodge decomposition: a method for solving boundary value problems*. Lecture notes in mathematics. Springer, 1995.
7. Clayton Shonkwiler. *Poincaré Duality Angles on Riemannian Manifolds with Boundary*. PhD thesis, University of Pennsylvania, 2009.
8. Clayton Shonkwiler. Poincaré duality angles and the Dirichlet-to-Neumann operator. *Inverse Problems*, 29(4), 2013.
9. Hermann von Helmholtz. Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. *Journal für die reine und angewandte Mathematik*, 55:25–55, 1858.
10. Max Wardetzky. *Discrete Differential Operators on Polyhedral Surfaces - Convergence and Approximation*. PhD thesis, Freie Universität Berlin, 2006.