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Discrete Mathematics for Bioinformatics (P1)

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Exercises 6

1. **Modulo Arithmetic**

   Prove the following theorem:
   For any positive intergers a and n, if $d = gcd(a, n)$ (the greatest common divisor of a and n), then

   $$\langle a \rangle = \langle d \rangle = \{0, d, 2d, \ldots, n - d\}$$

   and thus

   $$|\langle a \rangle| = n/d$$

   ($\langle a \rangle := \{a \cdot i \mod n \mid i \in \mathbb{N}\}$).

2. **Expected Value**

   The general definition of the expected value for a discrete random variable X with probability mass function $p(\mathtt{x})$ is given by:

   $$E(X) = \sum_i x_i p(x_i).$$

   Show that if random variable X only takes non-negative integral values the following formula is valid:

   $$E(X) = \sum_i P(X \geq i).$$

3. **Sum Formula**

   In the lecture we used the following relation:

   $$\sum_{j=0}^{k} \frac{j}{2^{j-1}} \leq 4$$

   Prove it! Hint: write the sum $\sum_{j=0}^{k} j \cdot x^{j-1}$ as a derivative of $\sum_{j=0}^{k} x^j$ and apply bounding.

4. **Independencies**

Random variables $(X_i)_{i\geq 1}$ are called *pairwise independent* if for all $1 \leq i < j$ and all r_i and r_j holds:

$$Pr(X_i = r_i \wedge X_j = r_j) = Pr(X_i = r_i) \cdot Pr(X_j = s_j)$$

Random variables are called *mutual independent* if for all $n \geq 2$ and all $1 \leq i_1 < i_2 < \ldots < i_n$ and $r_1, r_2, \ldots, r_n$ holds:

$$Pr(\bigwedge_{k=1}^{n} (X_{i_k} = r_k)) = \prod_{k=1}^{n} Pr(X_{i_k} = r_k)$$

(a) Let X and Y be random variables:

i. Prove that $E(X + Y) = E(X) + E(Y)$.

ii. Assume that X and Y are independent. Prove that $E(XY) = E(X)E(Y)$

(b) Given the sample space:

$$U = \{(123), (132), (213), (231), (312), (321), (111), (222), (333)\}$$

We choose a random element u in U. Let X_i the digit in u at position i (for $i = 1, 2, 3$), e.g. $X_3 = 2$ for $u = (312)$. Let N the random variable that equals X_2. Prove:

i. $\forall i, r : 1 \leq i \leq 3, 1 \leq r \leq 3$ gilt: $Pr(X_i = r) = \frac{1}{3}$.

ii. X_1, X_2, and X_3 pairwise independent.

iii. X_1, X_2, and X_3 are not mutual independent.

iv. $E(N) = 2$.

v. $\sum_{i=1}^{E(N)} E(X_i) = 4$.

vi. $E(\sum_{i=1}^{N} X_i) \neq \sum_{i=1}^{E(N)} E(X_i)$.