What is a Group Representation?

Giovanni De Gaetano

June 29, 2012

According to the Oxford Dictionary we have the following definition.

Representation*: the description or portrayal of someone or something in a particular way.*

In mathematical terms a representation of a group G is the description of the elements of G by matrices, or, more generally, the description of G as a subgroup of the automorphism group of a given object.

1 First Definitions

Definition 1 (Group Representation)**.** *Let G be a group. A (complex) G-representation is an homomorphism $\rho : G \rightarrow GL(V)$ where V is a vector space over $\mathbb{C}$.*

With abuse of notation we will denote the representation by V if the map ρ is understood from the context.

Observation 1. *As we suggested above we can consider more general representation using different groups in place of $GL(V)$. A commonly used object is $SL(n, \mathbb{Z})$, but any group of automorphisms could arise in this context.*

We can reformulate this definition in terms of group actions.

Observation 2. *The definition of group representation is equivalent to say that G acts on the $\mathbb{C}$-vector space V.*

Then we have associated a matrix to any element $g \in G$ in such a way that the product of elements in G translates into the usual matrix multiplication. Pay attention to the fact that we did not require the homomorphism ρ to be injective.

Example 1 (Trivial representation)**.** *Given any $\mathbb{C}$-vector space V and any group G we can define a representation $\rho : G \to GL(V)$ by $\rho(g) = id_V$ for any $g \in G$. It is called the trivial representation.*

Opposite to the trivial representation there is the notion of faithful representation.

Definition 2 (Faithful representation)**.** *The representation $\rho : G \to GL(V)$ is faithful if $Ker(\rho) = id_G$.*

Definition 3 (Finite representation)**.** *The representation ρ is finite if we require the vector space V to be finite. In this case we have $GL(V) = GL(dim(V), \mathbb{C})$.*

Observe that if G admits a finite faithful representation then it is a linear group. Indeed it is isomorphic to a subgroup of $GL(n, \mathbb{C})$ for some $n \in \mathbb{N}$.

Example 2 (Regular representation)**.** *To G we can associate the $\mathbb{C}$-vector space $G_{\mathbb{C}}$ which has basis $\{g : g \in G\}$. Then G acts on $G_{\mathbb{C}}$ by multiplication on the left. The induced representation is called regular representation of G.*

Using some additional or different algebraic structures on the set of matrices we can represent a variety of algebraic objects.

- Introducing the sum of matrices we turn $GL(n, \mathbb{C})$ into an associative algebra. Then we have the notion of associative algebra representation.
- Using the commutator $[M, N] = MN - NM$ instead of the usual matrix multiplication $GL(n, \mathbb{C})$ is a Lie algebra. Then we have the notion of Lie algebra representation.

Now that we have a new class of algebraic objects, i.e. group representations, we want to establish the notion of morphism between them.

Definition 4. *Given two representations V, W of G a morphism of G-representations is a $\mathbb{C}$-linear map $\varphi : V \to W$ which is invariant by the action of G. We denote by $Hom_G(V, W)$ the set of morphisms of G-representations.*

2 Reducibility of Representations

Definition 5 (Subrepresentations). *A G-subrepresentation of V is a vector subspace $W \subseteq V$ which is invariant by the action of G.*

Definition 6 (Irreducible representations). *A representation V of G is irreducible if it has no proper subrepresentations.*

Example 3 (Non irreducible representation). *Let S_3 act on $\mathbb{C}^3$ by permutation of the base elements. Then $\mathbb{C}^3$ is not an irreducible representation because of the proper subrepresentation $H = \{z(1,1,1) : z \in \mathbb{C}\}$.*

The usual operations on vector spaces translate into operations for representations. Given G-representations V, W the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are G-representations. Where the G-actions are given by $g(v \oplus w) = g(v) \oplus g(w)$ and $g(v \otimes w) = g(v) \otimes g(w)$. We can also construct the dual V^* by posing $g(\varphi) = \varphi \cdot g^{-1}$, and the exterior product, the symmetric product and so on.

Definition 7 (Reducible representations). *We say that the G-representation V is reducible if $V = W \oplus W'$ for some G-representations W and W'.*

A representation V is said to be reduced if it is not reducible.
Now it is natural to ask the following question.

Problem 1. *Is it always possible to reduce a given representation into the direct sum of irreducible representations?*

We answer it with a Theorem and a counterexample.

Theorem 1 (Complete Reducibility). *If G is a finite group then any G-representation V reduces to a direct sum of irreducible representations.*

We want to make here a small digression on the base field. It is clear that everything we did on $\mathbb{C}$ could have been done on any base field F, but here appears a first difference. Indeed if we work on a field F of positive characteristic p then the complete reducibility theorem holds under the further assumption that p does not divide the order of G.

Now we show a counterexample to the complete reducibiliy theorem in the case that G is not finite.

Example 4. *Let* $(\mathbb{R},+)$ *act on* $\mathbb{R}^2$ *by* $a((x,y)) = (x+ay, y)$*. The induced representation is reduced but not irreducible.*

Proof. We have a proper subrepresentation $R_1 = \mathbb{R}(1,0)$, where $\mathbb{R}$ acts trivially. This proves that our original representation is not irreducible.

Now we suppose that the subrepresentation R_1 has a complement $R_2 = \mathbb{R}(\gamma, \delta)$ for $\delta \neq 0$. R_2 must have dimension 1 as real vector space. Then acting with the element $a = \frac{-\gamma}{\delta}$ we deduce the form $R_2 = \mathbb{R}(0,\delta) = \mathbb{R}(0,1)$. But this is not an $(\mathbb{R},+)$-representation. □

Lemma 1 (Schur)**.** *Let* $\varphi : V \to W$ *be a morphism of irreducible* G*-representations. Then:*

1. $\varphi = 0$ *or* φ *is an isomorphism.*

2. *If* $V \simeq W$ *then* $\varphi = \lambda \cdot id_V$ *for* $\lambda \in \mathbb{C}$*.*

Exercise 1. *Prove Schur's Lemma.* [Hint: Kernel and images of $\mathbb{C}$-linear maps are vector subspaces.]

3 Characters

Throughout this section we assume G to be a finite group.

Definition 8 (Characters for finite groups)**.** *Given a* G*-representation* $\rho : G \to GL(V)$*, we define the associated character* $\chi_V : G \to \mathbb{C}$ *by* $\chi_V(g) = Tr(\rho(g))$*.*

Exercise 2 (Behavior of Characters)**.** *For* V, W *two generical representations of the group* G *holds:*

$$\chi_{V \oplus W} = \chi_V + \chi_W,$$

$$\chi_{V \otimes W} = \chi_V \cdot \chi_W,$$

$$\chi_{V^*} = \overline{\chi_V}.$$

Now we fix a general G-representation V.

Observation 3. *In general, for* $g \in G$*, the map* $g : V \to V$ *is not a morphism of* G*-representations.*

Proof. Indeed G in general is not commutative. □

But we can construct a morphism of G-representations in the following way, first we consider the set:

$$V^G = \{v \in V|\; gv = v \;\forall\; g \in G\}.$$

Which is clearly a G-subrepresentation of V. Then we define the map:

$$\varphi = \frac{1}{|G|}\sum_{g\in G} g : V \to V^G.$$

Exercise 3. *The map φ defined above is a morphism of G-representations and a projection.*

Since φ is a projection $dim_{\mathbb{C}} V^G = Tr(\varphi)$. And thanks to this last observation:

$$dim_{\mathbb{C}} V^G = Tr(\varphi) = \frac{1}{|G|}\sum_{g\in G} Tr(g) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g).$$

In specific the last sum must be zero if V is an irreducible representation.

Now, applying Schur Lemma to two irreducible representations V, W we get that $dim_{\mathbb{C}}(Hom_G(V,W))$ is 1 if $V = W$ and 0 otherwise. But, denoting by $Hom(V,W$ the set of $\mathbb{C}$-linear maps from V to W, we have that $Hom_G(V,W) = Hom(V,W)^G$ and $Hom(V,W) = V^* \otimes W$. By the exercise above:

$$\chi_{Hom(V,W)} = \overline{\chi_V} \cdot \chi_W.$$

And finally we get that the sum $\frac{1}{|G|}\sum_{g\in G} \overline{\chi_V(g)}\chi_W(g)$ is 1 if $V = W$ and 0 otherwise.

Inspired by the sum above we define a Hermitian product on the set of complex valued functions on G.

$$\langle \alpha, \beta \rangle := \frac{1}{|G|}\sum_{g\in G} \overline{\alpha(g)}\beta(g).$$

We have proved the next very important result.

Theorem 2. *The characters of irreducible representations are orthonormals with respect to the Hermitian product* $\langle , \rangle$ *defined above. In specific a representation* V *is irreducible if and only if* $\langle \chi_V, \chi_V \rangle = 1$.

The last statement is true because if it was $V = V_1 \oplus V_2$ then we would have $\langle \chi_V, \chi_V \rangle = \langle \chi_{V_1}, \chi_{V_1} \rangle + \langle \chi_{V_2}, \chi_{V_2} \rangle$. For the very same reason:

Corollary 1. *If* $V = V_1^{a_1} \oplus ... \oplus V_n^{a_n}$ *then* $a_i = \langle \chi_V, \chi_{V_i} \rangle$.

We can deduce a last nice result in a few steps.

Exercise 4 (Decomposition of the regular representation)**.** *Prove:*

1. $\chi_{G_{\mathbb{C}}}(g) = |G|$ *if and only if* $g = id_G$. *Otherwise it is zero.*
2. *Compute* $\chi_V(id_G)$ *for any representation* V.
3. *Deduce that every irreducible representation* V_i *appears in* $G_{\mathbb{C}}$ *exactly* $dimV_i$ *times.*

4 Relation with Fourier Analysis.

In this section we want to justify heuristically the following statement. Which somehow justifies a deeper study of Representation Theory.

Fourier Analysis is a special case of Representation Theory.

A formal statement can be made and proved, but it requires to work with topological locally compact commutative groups instead of finite groups (TLCCG for brevity, we assume our group G to be of this form for the whole section). In any case we will be able to translate some of the results and formulas of the last section.

Definition 9 (Characters for TLCCG)**.** *A character* χ *of* G *is a continuous morphism* $\chi : G \to T$. *Where* T *is the multiplicative group of complex numbers of modulus* 1.

It can be proved that the characters form a (topological) group $\tilde{G}$.

If we fix the additive group $G = \mathcal{S}^1$, we have the relation $\tilde{\mathcal{S}^1} \simeq \mathbb{Z}$. Where the characters are functions of the form e^{inx} for $n \in \mathbb{Z}$. It can be also shown that those characters correspond to irreducible representations of $\mathcal{S}^1$ as in the case of finite groups.

Now we can translate the Hermitian product given at the end of the last section into "continuous" terms. The size of the group $|G|$ became the lenght of $\mathcal{S}^1$. The discrete sum over elements of G corresponds to the integration over $\mathcal{S}^1$. The product then became:

$$\langle \alpha, \beta \rangle = \frac{1}{2\pi} \int_{\mathcal{S}^1} \overline{\alpha(x)} \beta(x) dx \sim \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

And our characters $\{e^{inx} | n \in \mathbb{N}\}$ are orthonormal with respect to this Hermitian product.

Fourier analysis does the very same thing, indeed it decomposes the Hilbert space of L^2-functions on $\mathcal{S}^1$ with respect to a suitable inner product. The inner product and the orthonormal basis he found in the late 19^{th} century are the same we found applying modern Representation Theory.

In a more abstract sense if we have a group G acting on the space X (we can think to G as a group of symmetries of G), then G acts also on the space $\mathcal{F}(X)$ of functions on X. If we know the irreducible representations of G then we can attempt to decompose $\mathcal{F}(X)$ into a direct sum of them, recovering in this way some information.

References

[1] Gaiffi, Sacco, Teoria delle Rappresentazioni, Appunti rivisitati.

[2] Joyal, Street, An Introduction to Tannaka Duality and Quantum Groups, *Category Theory, Lecture Notes in Math. 1991 vol. 1488 pp. 412-492.*