What is a Group Representation?

Giovanni De Gaetano

June 29, 2012

According to the Oxford Dictionary we have the following definition.

Representation: the description or portrayal of someone or something in a particular way.

In mathematical terms a representation of a group $G$ is the description of the elements of $G$ by matrices, or, more generally, the description of $G$ as a subgroup of the automorphism group of a given object.

1 First Definitions

**Definition 1** (Group Representation). Let $G$ be a group. A (complex) $G$-representation is an homomorphism $\rho : G \to GL(V)$ where $V$ is a vector space over $\mathbb{C}$.

With abuse of notation we will denote the representation by $V$ if the map $\rho$ is understood from the context.

**Observation 1.** As we suggested above we can consider more general representation using different groups in place of $GL(V)$. A commonly used object is $SL(n,\mathbb{Z})$, but any group of automorphisms could arise in this context.

We can reformulate this definition in terms of group actions.

**Observation 2.** The definition of group representation is equivalent to say that $G$ acts on the $\mathbb{C}$-vector space $V$. 
Then we have associated a matrix to any element \( g \in G \) in such a way that the product of elements in \( G \) translates into the usual matrix multiplication. Pay attention to the fact that we did not require the homomorphism \( \rho \) to be injective.

**Example 1** (Trivial representation). Given any \( \mathbb{C} \)-vector space \( V \) and any group \( G \) we can define a representation \( \rho : G \rightarrow GL(V) \) by \( \rho(g) = id_V \) for any \( g \in G \). It is called the trivial representation.

Opposite to the trivial representation there is the notion of faithful representation.

**Definition 2** (Faithful representation). The representation \( \rho : G \rightarrow GL(V) \) is faithful if \( \text{Ker}(\rho) = id_G \).

**Definition 3** (Finite representation). The representation \( \rho \) is finite if we require the vector space \( V \) to be finite. In this case we have \( GL(V) = GL(\text{dim}(V), \mathbb{C}) \).

Observe that if \( G \) admits a finite faithful representation then it is a linear group. Indeed it is isomorphic to a subgroup of \( GL(n, \mathbb{C}) \) for some \( n \in \mathbb{N} \).

**Example 2** (Regular representation). To \( G \) we can associate the \( \mathbb{C} \)-vector space \( G_\mathbb{C} \) which has basis \( \{ g : g \in G \} \). Then \( G \) acts on \( G_\mathbb{C} \) by multiplication on the left. The induced representation is called regular representation of \( G \).

Using some additional or different algebraic structures on the set of matrices we can represent a variety of algebraic objects.

- Introducing the sum of matrices we turn \( GL(n, \mathbb{C}) \) into an associative algebra. Then we have the notion of associative algebra representation.

- Using the commutator \([M, N] = MN - NM\) instead of the usual matrix multiplication \( GL(n, \mathbb{C}) \) is a Lie algebra. Then we have the notion of Lie algebra representation.

Now that we have a new class of algebraic objects, i.e. group representations, we want to establish the notion of morphism between them.

**Definition 4.** Given two representations \( V, W \) of \( G \) a morphism of \( G \)-representations is a \( \mathbb{C} \)-linear map \( \varphi : V \rightarrow W \) which is invariant by the action of \( G \). We denote by \( \text{Hom}_G(V, W) \) the set of morphisms of \( G \)-representations.
2 Reducibility of Representations

Definition 5 (Subrepresentations). A $G$-subrepresentation of $V$ is a vector subspace $W \subseteq V$ which is invariant by the action of $G$.

Definition 6 (Irreducible representations). A representation $V$ of $G$ is irreducible if it has no proper subrepresentations.

Example 3 (Non irreducible representation). Let $S_3$ act on $\mathbb{C}^3$ by permutation of the base elements. Then $\mathbb{C}^3$ is not an irreducible representation because of the proper subrepresentation $H = \{z(1,1,1) : z \in \mathbb{C}\}$.

The usual operations on vector spaces translate into operations for representations. Given $G$-representations $V, W$ the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are $G$-representations. Where the $G$-actions are given by $g(v \oplus w) = g(v) \oplus g(w)$ and $g(v \otimes w) = g(v) \otimes g(w)$. We can also construct the dual $V^*$ by posing $g(\varphi) = \varphi \cdot g^{-1}$, and the exterior product, the symmetric product and so on.

Definition 7 (Reducible representations). We say that the $G$-representation $V$ is reducible if $V = W \oplus W'$ for some $G$-representations $W$ and $W'$.

A representation $V$ is said to be reduced if it is not reducible.

Now it is natural to ask the following question.

Problem 1. Is it always possible to reduce a given representation into the direct sum of irreducible representations?

We answer it with a Theorem and a counterexample.

Theorem 1 (Complete Reducibility). If $G$ is a finite group then any $G$-representation $V$ reduces to a direct sum of irreducible representations.

We want to make here a small digression on the base field. It is clear that everything we did on $\mathbb{C}$ could have been done on any base field $F$, but here appears a first difference. Indeed if we work on a field $F$ of positive characteristic $p$ then the complete reducibility theorem holds under the further assumption that $p$ does not divide the order of $G$.

Now we show a counterexample to the complete reducibility theorem in the case that $G$ is not finite.
Example 4. Let \((\mathbb{R}, +)\) act on \(\mathbb{R}^2\) by \(a((x, y)) = (x + ay, y)\). The induced representation is reduced but not irreducible.

**Proof.** We have a proper subrepresentation \(R_1 = \mathbb{R}(1, 0)\), where \(\mathbb{R}\) acts trivially. This proves that our original representation is not irreducible.

Now we suppose that the subrepresentation \(R_1\) has a complement \(R_2 = \mathbb{R}(\gamma, \delta)\) for \(\delta \neq 0\). \(R_2\) must have dimension 1 as real vector space. Then acting with the element \(a = \frac{-\gamma}{\delta}\) we deduce the form \(R_2 = \mathbb{R}(0, \delta) = \mathbb{R}(0, 1)\). But this is not an \((\mathbb{R}, +)\)-representation.

**Lemma 1** (Schur). Let \(\varphi : V \rightarrow W\) be a morphism of irreducible \(G\)-representations. Then:

1. \(\varphi = 0\) or \(\varphi\) is an isomorphism.
2. If \(V \cong W\) then \(\varphi = \lambda \cdot \text{id}_V\) for \(\lambda \in \mathbb{C}\).

**Exercise 1.** Prove Schur’s Lemma. [Hint: Kernel and images of \(\mathbb{C}\)-linear maps are vector subspaces.]

3 Characters

Throughout this section we assume \(G\) to be a finite group.

**Definition 8** (Characters for finite groups). Given a \(G\)-representation \(\rho : G \rightarrow GL(V)\), we define the associated character \(\chi_V : G \rightarrow \mathbb{C}\) by \(\chi_V(g) = Tr(\rho(g))\).

**Exercise 2** (Behavior of Characters). For \(V, W\) two general representations of the group \(G\) holds:

\[
\chi_{V \oplus W} = \chi_V + \chi_W,
\]
\[
\chi_{V \otimes W} = \chi_V \cdot \chi_W,
\]
\[
\chi_V^* = \overline{\chi_V}.
\]

Now we fix a general \(G\)-representation \(V\).

**Observation 3.** In general, for \(g \in G\), the map \(g : V \rightarrow V\) is not a morphism of \(G\)-representations.

**Proof.** Indeed \(G\) in general is not commutative.
But we can construct a morphism of $G$-representations in the following way, first we consider the set:

$$V^G = \{ v \in V \mid gv = v \ \forall \ g \in G \}. $$

Which is clearly a $G$-subrepresentation of $V$. Then we define the map:

$$\varphi = \frac{1}{|G|} \sum_{g \in G} g : V \to V^G. $$

**Exercise 3.** The map $\varphi$ defined above is a morphism of $G$-representations and a projection.

Since $\varphi$ is a projection $\dim \mathbb{C} V^G = Tr(\varphi)$. And thanks to this last observation:

$$\dim \mathbb{C} V^G = Tr(\varphi) = \frac{1}{|G|} \sum_{g \in G} Tr(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g). $$

In specific the last sum must be zero if $V$ is an irreducible representation.

Now, applying Schur Lemma to two irreducible representations $V,W$ we get that $\dim \mathbb{C}(\text{Hom}_G(V,W))$ is 1 if $V = W$ and 0 otherwise. But, denoting by $\text{Hom}(V,W)$ the set of $\mathbb{C}$-linear maps from $V$ to $W$, we have that $\text{Hom}_G(V,W) = \text{Hom}(V,W)^G$ and $\text{Hom}(V,W) = V^* \otimes W$. By the exercise above:

$$\chi_{\text{Hom}(V,W)} = \overline{\chi_V} \cdot \chi_W. $$

And finally we get that the sum $\frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g)$ is 1 if $V = W$ and 0 otherwise.

Inspired by the sum above we define a Hermitian product on the set of complex valued functions on $G$.

$$\langle \alpha, \beta \rangle := \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g). $$

We have proved the next very important result.
Theorem 2. The characters of irreducible representations are orthonormals with respect to the Hermitian product $\langle, \rangle$ defined above. In specific a representation $V$ is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

The last statement is true because if it was $V = V_1 \oplus V_2$ then we would have $\langle \chi_V, \chi_V \rangle = \langle \chi_{V_1}, \chi_{V_1} \rangle + \langle \chi_{V_2}, \chi_{V_2} \rangle$. For the very same reason:

Corollary 1. If $V = V_1^{a_1} \oplus ... \oplus V_n^{a_n}$ then $a_i = \langle \chi_V, \chi_{V_i} \rangle$.

We can deduce a last nice result in a few steps.

Exercise 4 (Decomposition of the regular representation). Prove:

1. $\chi_{G_C}(g) = |G|$ if and only if $g = id_G$. Otherwise it is zero.
2. Compute $\chi_V(id_G)$ for any representation $V$.
3. Deduce that every irreducible representation $V_i$ appears in $G_C$ exactly $\dim V_i$ times.

4 Relation with Fourier Analysis.

In this section we want to justify heuristically the following statement. Which somehow justifies a deeper study of Representation Theory.

"Fourier Analysis is a special case of Representation Theory."

A formal statement can be made and proved, but it requires to work with topological locally compact commutative groups instead of finite groups (TLCCG for brevity, we assume our group $G$ to be of this form for the whole section). In any case we will be able to translate some of the results and formulas of the last section.

Definition 9 (Characters for TLCCG). A character $\chi$ of $G$ is a continuous morphism $\chi : G \to T$. Where $T$ is the multiplicative group of complex numbers of modulus 1.

It can be proved that the characters form a (topological) group $\tilde{G}$.

If we fix the additive group $G = S^1$, we have the relation $\tilde{S}^1 \simeq \mathbb{Z}$. Where the characters are functions of the form $e^{inx}$ for $n \in \mathbb{Z}$. It can be also shown
that those characters correspond to irreducible representations of $S^1$ as in the case of finite groups.

Now we can translate the Hermitian product given at the end of the last section into "continuous" terms. The size of the group $|G|$ became the length of $S^1$. The discrete sum over elements of $G$ corresponds to the integration over $S^1$. The product then became:

$$\langle \alpha, \beta \rangle = \frac{1}{2\pi} \int_{S^1} \overline{\alpha(x)} \beta(x) dx \sim \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

And our characters $\{e^{inx} | n \in \mathbb{N} \}$ are orthonormal with respect to this Hermitian product.

Fourier analysis does the very same thing, indeed it decomposes the Hilbert space of $L^2$-functions on $S^1$ with respect to a suitable inner product. The inner product and the orthonormal basis he found in the late 19th century are the same we found applying modern Representation Theory.

In a more abstract sense if we have a group $G$ acting on the space $X$ (we can think to $G$ as a group of symmetries of $G$), then $G$ acts also on the space $\mathcal{F}(X)$ of functions on $X$. If we know the irreducible representations of $G$ then we can attempt to decompose $\mathcal{F}(X)$ into a direct sum of them, recovering in this way some information.

References
