What are... Catalan numbers?

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BMS
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Triangulations of a $n$-gon

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<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
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Finally, Euler gave the following formula:

$$\frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n - 10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot (n - 1)}$$

which is now called the $(n - 2)$nd **Catalan** number.
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This number can be rewritten as

\[
C_n = \frac{1}{n + 1} \binom{2n}{n}.
\]
In 1758, Johann Segner gave a recurrence formula answering Euler’s problem:

\[ C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}. \]
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Then, Euler essentially solved the recurrence though without giving a complete proof.
Dissection of a \( n \)-gon

Problem (Johann Pfaff & Nicolaus Fuss (1791))

Let \( n, k \in \mathbb{N} \). How many dissections of a \( (kn + 2) \)-gon using \( (k + 2) \)-gons are there?

In 1791, Niklaus Fuss gave an answer using Segner's recurrence formula:

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C_{n,k} = \frac{1}{n} \binom{n}{k+1} 
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These numbers are now known as Fuss-Catalan numbers.

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Further developments

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- Gabriel Lamé finally gave a complete proof of Euler-Segner formula;
- Eugène Charles Catalan further discussed on this subject;
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Liouville mentioned that Lamé was the first one to give such an elegant solution.

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Numerous Catalan structures

During the 20\textsuperscript{th} century, many different objects were revealed to be enumerated by Catalan numbers:

- M. Kuchinski found 31 structures and 158 bijections between them (PhD thesis, 1977);
- R. Stanley counts 190 structures counted by Catalan numbers (as of 21/08/2010).

For example:
- Dyck paths (path from (0,0) to (n,n) always above diagonal, using East and North steps);
- Plane trees with n + 1 vertices;
- Dimension of the space of invariants of $\text{SL}(2, \mathbb{C})$ acting on the $2^n$-th tensor power $T_{2^n}(V)$, of its two-dimensional representation $V$;
- Standard Young tableaux of shape $(n, n-1)$;
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Bijective proof using triangulations

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\[(n + 2)\text{-gon} \quad \text{and} \quad (n + 3)\text{-gon}\]
A simple geometric proof

Bijective proof using triangulations

\[(n + 2)\text{-gon}\]

\(C_n\) objects

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\(C_{n+1}\) objects
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\[ C_n \text{ objects} \]

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\[(n + 2)\text{-gon}\]
\[\text{(4n + 2)}C_n \text{ objects}\]

\[(n + 3)\text{-gon}\]
\[C_{n+1} \text{ objects}\]
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\[(4n + 2)C_n \text{ objects} \quad (n + 2)C_{n+1} \text{ objects} \]
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\[(n + 2)\text{-gon}\]
\[(4n + 2)C_n \text{ objects}\]

\[(n + 3)\text{-gon}\]
\[(n + 2)C_{n+1} \text{ objects}\]
So, we have the following relation

\[ C_{n+1} = C_n (4n + 2)(n + 2) \]

with \( C_1 = 1 \), we get the binomial formula

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CQFD
A more complicated example

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