Linear programming

Optimization Problems

- **General optimization problem**

  \[
  \max \{ z(x) \mid f_j(x) \leq 0, x \in D \} \text{ or } \min \{ z(x) \mid f_j(x) \leq 0, x \in D \}
  \]
  where \( D \subseteq \mathbb{R}^n \), \( f_j : D \to \mathbb{R} \), for \( j = 1, \ldots, m \), \( z : D \to \mathbb{R} \).

- **Linear optimization problem**

  \[
  \max \{ c^T x \mid Ax \leq b, x \in \mathbb{R}^n \}, \text{ with } c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m
  \]

- **Integer optimization problem**: \( x \in \mathbb{Z}^n \)

- **0-1 optimization problem**: \( x \in \{0, 1\}^n \)

Feasible and optimal solutions

- Consider the optimization problem

  \[
  \max \{ z(x) \mid f_j(x) \leq 0, x \in D, j = 1, \ldots, m \}
  \]

- A **feasible solution** is a vector \( x' \in D \subseteq \mathbb{R}^n \) such that \( f_j(x') \leq 0 \), for all \( j = 1, \ldots, m \).

- The set of all feasible solutions is called the **feasible region**.

- An **optimal solution** \( x^* \) is a feasible solution such that

  \[
  z(x^*) = \max \{ z(x) \mid f_j(x) \leq 0, x \in D, j = 1, \ldots, m \}.
  \]

- Feasible or optimal solutions

  - need not exist,
  - need not be unique.

Transformations

- \( \min \{ z(x) \mid x \in S \} = \max \{ -z(x) \mid x \in S \} \).

- \( f(x) \geq a \) if and only if \( -f(x) \leq -a \).

- \( f(x) = a \) if and only if \( f(x) \leq a \land -f(x) \leq -a \).

Lemma

Any linear programming problem can be brought to the form

\[
\max \{ c^T x \mid Ax \leq b \} \text{ or } \max \{ c^T x \mid Ax = b, x \geq 0 \}.
\]

**Proof:**

- a) \( a^T x \leq \beta \quad \sim \quad a^T x + x' = \beta, x' \geq 0 \) (slack variable)

- b) \( x \) free \( \sim \quad x = x^+ - x^-, x^+, x^- \geq 0 \).
Practical problem solving

1. Model building
2. Model solving
3. Model analysis

Example: Production problem

A firm produces $n$ different goods using $m$ different raw materials.

- $b_i$: available amount of the $i$-th raw material
- $a_{ij}$: number of units of the $i$-th material needed to produce one unit of the $j$-th good
- $c_j$: revenue for one unit of the $j$-th good.

Decide how much of each good to produce in order to maximize the total revenue \( \rightarrow \) decision variables $x_j$.

Linear programming formulation

\[
\begin{align*}
\text{max} & \quad c_1 x_1 + \cdots + c_n x_n \\
\text{w.r.t.} & \quad a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1, \\
& \quad \vdots \\
& \quad a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m, \\
& \quad x_1, \ldots, x_n \geq 0.
\end{align*}
\]

In matrix notation:

\[
\text{max} \{ c^T x \mid Ax \leq b, x \geq 0 \},
\]

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, x \in \mathbb{R}^n$.

Geometric illustration

\[
\begin{align*}
\text{max} & \quad x_1 + x_2 \\
\text{w.r.t.} & \quad x_1 + 2x_2 \leq 3 \\
& \quad 2x_1 + x_2 \leq 3 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Polyhedra

- **Hyperplane** $H = \{ x \in \mathbb{R}^n \mid a^T x = \beta \}$, $a \in \mathbb{R}^n \setminus \{0\}, \beta \in \mathbb{R}$
- **Closed halfspace** $\overline{H} = \{ x \in \mathbb{R}^n \mid a^T x \leq \beta \}$
- **Polyhedron** $P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$, $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$
- **Polytope** $P = \{ x \in \mathbb{R}^n \mid Ax \leq b, l \leq x \leq u \}$

The feasible set
$$ P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$$
of a linear optimization problem is a polyhedron.

Vertices, Faces, Facets

- $P \subseteq \overline{H}, H \cap P \neq \emptyset$ (Supporting hyperplane)
- $F = P \cap H$ (Face)
- $\dim(F) = 0$ (Vertex)
- $\dim(F) = 1$ (Edge)
- $\dim(F) = \dim(P) - 1$ (Facet)
- $P$ pointed: $P$ has at least one vertex.

Illustration

Simplex Algorithm: Geometric view

**Linear optimization problem**
$$ \max \{ c^T x \mid Ax \leq b, x \in \mathbb{R}^n \} \quad (LP) $$

Simplex-Algorithm (Dantzig 1947)
1. Find a vertex of $P$.

2. Proceed from vertex to vertex along edges of $P$ such that the objective function $z = c^T x$ increases.

3. Either a vertex will be reached that is optimal, or an edge will be chosen which goes off to infinity and along which $z$ is unbounded.