Duality (2)

Theorem. (Duality Theorem)
If the primal has an optimal solution $x^*$ then the dual has an optimal solution $y^*$ such that

$$c^T x^* = y^*^T b$$

Proof. (Constructive! $\rightarrow$ blackboard)

Some properties:

- The dual of the dual is the primal (exercise).
- Corollary: Primal has an optimal solution \textit{iff} dual has an optimal solution.
- Corollary: Primal unbounded $\rightarrow$ dual infeasible.
- Corollary: Dual unbounded $\rightarrow$ primal infeasible.
- Primal and dual can both be infeasible (Example).
Practical implications:

- It might be faster to run the Simplex algorithm on the dual.
- Duality provides an elegant way of *proving optimality*. So, solutions come with a certificate, which is always good.
Simplex Algorithm: Geometric view

- Hyperplane $H = \{x \in \mathbb{R}^n \mid a^T x = \beta\}, a \in \mathbb{R}^n \setminus \{0\}, \beta \in \mathbb{R}$

- Closed halfspace $\overline{H} = \{x \in \mathbb{R}^n \mid a^T x \leq \beta\}$

- Polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

- Polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b, \quad l \leq x \leq u\}$

The feasible set

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

of a linear optimization problem is a polyhedron.
Simplex Algorithm: Geometric view

- $P \subseteq \overline{H}$, $H \cap P \neq \emptyset$ (Supporting hyperplane)
- $F = P \cap H$ (Face)
- $\dim(F) = 0$ (Vertex)
- $\dim(F) = 1$ (Edge)
- $\dim(F) = \dim(P) - 1$ (Facet)
Simplex Algorithm: Geometric view
Linear optimization problem

\[
\max \{ c^T x \mid Ax \leq b, x \in \mathbb{R}^n, x \geq 0 \}
\]  

(LP)

Simplex-Algorithm (Dantzig 1947)

1. Find a vertex of \( P \).

2. Proceed from vertex to vertex along edges of \( P \) such that the objective function \( z = c^T x \) increases.

3. Either a vertex will be reached that is optimal, or an edge will be chosen which goes off to infinity and along which \( z \) is unbounded.