Linear Algebra

We recall some basic definitions from linear algebra.

- A vector \(x \in \mathbb{R}^n\) is a linear combination of the vectors \(x^1, \ldots, x^p \in \mathbb{R}^n\) if
  \[
  x = \sum_{i=1}^{p} \lambda_i x^i, \text{ for some } \lambda_1, \ldots, \lambda_p \in \mathbb{R}
  \]

- If in addition, \(\lambda_1, \ldots, \lambda_p = 1\), \(x\) is an affine combination of \(x^1, \ldots, x^p\).

- For \(X \subseteq \mathbb{R}^n, X \neq \emptyset\), the linear (resp. affine) hull of \(X\), denoted by \(\text{lin}(X)\) (resp. \(\text{aff}(X)\)), is the set of all linear (resp. affine) combinations of finitely many vectors of \(X\).

- A set \(X \subseteq \mathbb{R}^n, X \neq \emptyset\), is called linearly independent (resp, affinely independent) if no vector \(x \in X\) is expressible as a linear (resp. affine) combination of the vectors in \(X \setminus \{x\}\), otherwise \(X\) is called linearly dependent (resp. affinely dependent).

- The rank (resp affine rank) of \(X\), denoted by \(\text{rank}(X)\) (resp \(\text{arank}(X)\)), is the cardinality of the largest linearly (resp. affinely) independent subset of \(X\).

- If \(0 \in \text{aff}(X)\) then \(\text{arank}(X) = \text{rank}(X) + 1\), otherwise \(\text{arank}(X) = \text{rank}(X)\).

- By definition, the dimension of a set \(X \subseteq \mathbb{R}^n\) is the maximum number of affinely independent vectors in \(X\) minus one, i.e., \(\dim(X) = \text{arank}(X) - 1\).

- If \(0 \in \text{aff}(X)\) then \(\dim(X) = \text{rank}(X)\), otherwise \(\dim(X) = \text{rank}(X) - 1\).

Example:

Let
\[
P = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1\},
\]
\[
F_1 = P \cap \{x \in \mathbb{R}^2 \mid x_1 = 0\},
\]
\[
F_2 = P \cap \{x \in \mathbb{R}^2 \mid x_1 = 1\}.
\]
We have

\[ \text{aff}(P) = \mathbb{R}^2, \]
\[ \text{aff}(F_1) = \{ x \in \mathbb{R}^2 \mid x_1 = 0 \}, \]
\[ \text{aff}(F_2) = \{ x \in \mathbb{R}^2 \mid x_1 = 1 \}. \]

- We can easily see that the vectors \((1, 0)^T \in P\) and \((0, 1)^T \in P\) are linearly independent in \(P\). Therefore, \(\text{rank}(P) = 2\). Since \(0 \in \text{aff}(P)\), we have \(\text{dim}(P) = \text{rank}(P) = 2\) (\(P\) is full-dimensional).

- \(F_1\) and \(F_2\) are two faces of \(P\) defined by the valid inequalities \(0 \leq x_1\) and \(x \leq 1\), respectively.

- No other vector in \(F_1\) is linearly independent of the vector \((0, 1)^T \in F_1\). Then, \(\text{rank}(F_1) = 1\). Since \(0 \in \text{aff}(F_1)\), we have \(\text{dim}(F_1) = \text{rank}(F_1) = 1\).

- We can easily see that the vectors \((1, 0)^T \in F_2\) and \((1, 1)^T \in F_2\) are linearly independent in \(F_2\). Therefore, \(\text{rank}(F_2) = 2\). Since \(0 \notin \text{aff}(F_2)\), we have \(\text{dim}(F_2) = \text{rank}(F_2) - 1 = 1\).

**Hint:**

For a given set of vectors \(x^1, \ldots, x^p \in \mathbb{R}^n\), they are linearly independent if

\[ \lambda_1 x^1 + \lambda_2 x^2 + \ldots, + \lambda_p x^p = 0 \]

only when \(\lambda_1 = \lambda_2 = \ldots = \lambda_p = 0\).