3. n-step transition and stationarity

3.1 Def (X_n) MC with transition matrix $\mathbb{P} = (p_{ij})_{i,j \in E}$

The matrix $\mathbb{P}^{(n)} = (p_{ij}^{(n)})_{i,j \in E}$ with $p_{ij}^{(n)} = \sum_{i_1, \dots, i_{n-1} \in E} p_{i i_1} \cdots p_{i_{n-1} j}$

is called n-step transition matrix of (X_n).

3.2 Proposition With $\mathbb{P}^{(0)} := I$, I $\ell\times\ell$ identity matrix, we have

$\mathbb{P}^{(n)} = \mathbb{P}^n$ and $(*)$ $\mathbb{P}^{(n+m)} = \mathbb{P}^{(n)} \mathbb{P}^{(m)}$ f.a. $n, m \in \mathbb{N}_0$.

3.3 Remark

$(*)$ in 3.2 is called Chapman-Kolmogorov equation.

3.4 Ex For $\mathbb{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we have $\mathbb{P}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{P}^{2k}$, $\mathbb{P}^{2k+1} = \mathbb{P}$ f.a. $k \in \mathbb{N}$

3.5 Theorem α initial distribution of (X_n)

Then $\alpha_n \in [0,1]^\ell$ defined by $(\alpha_n)_i = P(X_n = i)$ is given

by $\alpha_n^T = \alpha^T \mathbb{P}^n$.

α_n represents the distribution after n steps.

3.6 Def

A probability distribution π satisfying $\pi^T = \pi^T \mathbb{P}$ is called

stationary distribution of $\mathbb{P}$ (or of the MC (X_n)).

3.7 Remark

1. We have $\pi^T \mathbb{P}^n = \pi^T$ f.a. $n \in \mathbb{N}$.
2. More to stationary distributions later (existence / uniqueness).

4. Communication and Reducibility

GA (X_n) MC on state space E with trans. matrix $\mathbb{P} = (p_{ij})_{i,j \in E}$

and initial distribution α.

4.1 Def Let $i, j \in E$.

Then j is accessible from i if $p_{ij}^{(n)} > 0$ for some $n \in \mathbb{N}_0$,

where $\mathbb{P}^{(0)} = I$.

Notation $i \to j$

4.2 Remark By definition $i \to i$, even if $p_{ii}^{(n)} = 0$ f.a. $n \geq 1$.

4.3 Theorem Let $i \in E$ s.t. $P(X_0 = i) > 0$.

Then $j \in E$ is accessible from i iff $P(\tau_j < \infty \mid X_0 = i) > 0$,

where $\tau_j : \Omega \to \mathbb{N}_0$ is a random variable with $\tau_j := \min\{n \geq 0 \mid X_n = j\}$
and $\tau_j = \infty$ if $X_n \neq j$ f.a. $n \geq 0$.

4.4 Remark

τ_j is called the hitting time of j.

If we consider $n > 0$ in the def., i.e., $T_j := \min\{n > 0 \mid X_n = j\}$, we call T_j return time to j.

4.5 Def Let $i, j \in E$

States i and j are said to communicate (notation $i \leftrightarrow j$) if $i \to j$ and $j \to i$.

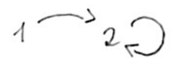
4.6 Prop./Def

"$\leftrightarrow$" is an equivalence relation on the state space E.

The equivalence classes partitioning E are called communication classes.

4.7 Ex - two-state MCs

1 comm. class : $\{1, 2\}$



2 comm. classes : $\{1\}, \{2\}$

4.8 Def

If there exists only one communication class, then the MC is called irreducible.

4.9 Ex

4.10 Def

A non-empty set $C \subseteq E$ is called closed, if $\sum_{j \in C} p_{ij} = 1$ f.a. $i \in C$.

A state $j \in E$ is absorbing if $\{j\}$ is closed.

4.11 Remark

Closed subsets of E can be analyzed as isolated systems thus reducing complexity.

5. Periodicity

5.1 Def Let $i \in E$.

Then $d_i := \gcd\{n \geq 1 \mid p_{ii}^{(n)} > 0\}$ (with gcd denoting the greatest common divisor) is called period of i. We set $d_i := \infty$ if $p_{ii}^{(n)} = 0$ f.a. $n \geq 1$.

If $d_i = 1$ then i is called aperiodic.

5.2 Ex - random walk

5.3 Remark - period d_i vs return to i

5.4 Theorem

Let $i, j \in E$ with $i \leftrightarrow j$. Then $d_i = d_j$.

5.5 Corollary Let (X_n) be irreducible.

Then all states of (X_n) have the same period, which is then called period of (X_n).

5.6 Theorem Let (X_n) be irreducible

Then there exists a unique partition of E into d classes $C_0, C_1, \ldots, C_{d-1} \subseteq E$ such that f.a. $k \in \{0, \ldots, d-1\}$, $i \in C_k$:

$$\sum_{j \in C_{k+1}} p_{ij} = 1 \quad , \quad \text{with } C_d := C_0 \text{ by convention}$$

and d maximal (i.e. there is no other such partition with more than d classes)

Moreover, d is the period of (X_n).

The C_k are called cyclic classes.

5.7 Ex random walk with two cyclic classes

5.8 Remarks

1. 5.6 does not hold if (X_n) is not irreducible.
2. period of (X_n) vs return to cyclic classes