3. n-step transition and stationarity

3.1 Def. \((X_n)\) MC with transition matrix \(P = (p_{ij})_{i,j \in E}\)

The matrix \(P^{(n)} = (p_{ij}^{(n)})_{i,j \in E}\) with \(p_{ij}^{(n)} = \sum_{i_1, \ldots, i_n \in E} p_{ii_1} \cdots p_{i_{n-1} i}\)

is called n-step transition matrix of \((X_n)\).

3.2 Proposition. With \(P^{(0)} = I\), the identity matrix, we have \(P^{(n)} = P^n\) and (*) \(P^{(n+m)} = P^{(n)} P^{(m)}\) if \(n, m \in \mathbb{N}\).

3.3 Remark. (*) in 3.2 is called Chapman-Kolmogorov equation.

3.4 Ex. For \(P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) we have \(P^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) and \(P^{2n+1} = P\) for \(n \in \mathbb{N}\).

3.5 Theorem. Let initial distribution \(\mu\) of \((X_n)\)

Then \(\mu \in [0,1]^E\) defined by \((\mu n)_i = P(X_n = i)\) is given by \(\mu^n = \mu^T P^n\).

\(\mu^n\) represents the distribution after \(n\) steps.

3.6 Def. A probability distribution \(\pi\) satisfying \(\pi^T = \pi^T P\) is called stationary distribution of \(P\) (or of the MC \((X_n)\)).

3.7 Remark.

1. We have \(\pi^T P^n = \pi^T\) if \(n \in \mathbb{N}\).

2. More to stationary distributions later (existence / uniqueness).

4. Communication and Reducibility

\((X_n)\) MC on state space \(E\) with transition matrix \(P = (p_{ij})_{i,j \in E}\) and initial distribution \(\mu\).

4.1 Def. Let \(i, j \in E\).

Then \(j\) is accessible from \(i\) if \(p_{ij}^{(n)} > 0\) for some \(n \in \mathbb{N}_0\), where \(P^{(0)} = I\).

Notation. \(i \rightarrow j\)

4.2 Remark. By definition \(i \rightarrow i\), even if \(p_{ii}^{(n)} = 0\) if \(n \geq 1\).
4.3 Theorem Let $i \in E$ s.t. $P(X_0 = i) > 0$. Then $j \in E$ is accessible from $i$ iff $P(\tau_j < \infty | X_0 = i) > 0$, where $\tau_j : \Omega \to [0, \infty]$ is a random variable with $\tau_j := \min \{ n \geq 0 | X_n = j \}$ and $\tau_j = \infty$ if $X_n \neq j$ s.a. $n \geq 0$.

4.4 Remark

$\tau_j$ is called the hitting time of $j$.

If we consider $n > 0$ in the def., i.e., $T_j := \min \{ n > 0 | X_n = j \}$, we call $T_j$ return time to $j$.

4.5 Def Let $i,j \in E$

States $i$ and $j$ are said to communicate (notation $i \leftrightarrow j$) if $i \rightarrow j$ and $j \rightarrow i$.

4.6 Prop/Def

"$\leftrightarrow$" is an equivalence relation on the state space $E$.

The equivalence classes partitioning $E$ are called communication classes.

4.7 Ex - two-state MCs

1 \[ \rightarrow \] 2

1 comm. class : \{1, 2\}

4.8 Def

If there exists only one communication class, then the MC is called irreducible.

4.9 Ex

4.10 Def

A non-empty set $C \subseteq E$ is called closed, if $i \in C \Rightarrow P(i,j) = 1$ s.a. $j \in C$.

A state $j \in E$ is absorbing if $\tau_j$ is closed.

4.11 Remark

Closed subsets of $E$ can be analyzed as isolated systems thus reducing complexity.
5. Periodicity

5.1 Definition

Let \( i \in E \).
Then \( d_i = \text{gcd} \{ n \geq 1 \mid P_i^{(n)} > 0 \} \) (with \( \text{gcd} \) denoting the greatest common divisor) is called the period of \( i \). We set \( d_i = \infty \) if \( P_i^{(n)} = 0 \) \( \forall \) \( n \geq 1 \).

If \( d_i = 1 \) then \( i \) is called aperiodic.

5.2 Example - random walk

5.3 Remark

period \( d_i \) vs return to \( i \)

5.4 Theorem

Let \( i, j \in E \) with \( i \rightarrow j \). Then \( d_i = d_j \).

5.5 Corollary

Let \( (X_n) \) be irreducible.
Then all states of \( (X_n) \) have the same period, which is then called the period of \( (X_n) \).

5.6 Theorem

Let \( (X_n) \) be irreducible.
Then there exists a unique partition of \( E \) into \( d \) classes \( C_0, C_1, \ldots, C_{d-1} \in E \), such that \( \forall i, j \in [0, \ldots, d-1], i \in C_k : \)

\[ P_{ij} = 1 \]

with \( C_d := C_0 \) by convention.

and \( d \) maximal (i.e. there is no other such partition with more than \( d \) classes).
Moreover, \( d \) is the period of \( (X_n) \).
The \( C_k \) are called cyclic classes.

5.7 Example - random walk with two cyclic classes

5.8 Remarks

1. 5.6 does not hold if \( (X_n) \) is not irreducible.
2. period of \( (X_n) \) vs return to cyclic classes.