1. Basics

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $E = \{1, 2, \ldots, i, \ldots\}$, a finite set (or $E$ a countable set), and $(X_n)_{n \in \mathbb{N}_0}$ a sequence of random variables $X_n : \Omega \to E$, i.e., $\mathbb{N}_0$.

1.1 Def. $(X_n)_{n \in \mathbb{N}_0}$ is called discrete-time stochastic process with state space $E$.

If $X_n = i$, $i \in E$, $n \in \mathbb{N}_0$, the process is said to be in state $i$ at time $n$.

For $\omega \in \Omega$, the $E$-valued sequence $(X_0(\omega), X_1(\omega), X_2(\omega), \ldots)$ is called realization (or trajectory or sample path) associated with $\omega$.

1.2 Remarks

1. Recall the notion of distribution $\mu : E \to \mathbb{R}$, $i \mapsto P(X = i)$ for random variable $X : \Omega \to E$.

Notation: $P(X = i) = P(\{X = i\}) = P(\{\omega \in \Omega \mid X(\omega) = i\})$

1.3 Def.

A discrete-time stochastic process $(X_n)_{n \in \mathbb{N}_0}$ is called Markov chain if the Markov property

\[ P(X_n = i_n \mid X_{n-1} = i_{n-1}, \ldots, X_1 = i_1) = P(X_n = i_n \mid X_{n-1} = i_{n-1}) \]

holds a.s. for $n, i_n, \ldots, i_1 \in E$ (assuming both sides of eq. (4) are defined).

If $P_{ij}(n) = P(X_n = j \mid X_{n-1} = i)$ does not depend on $n$, then the MC is called (time) homogeneous.

1.4 Remarks

The conditional probability $P(A \mid B)$ is not defined if $P(B) = 0$.

In the following consider statements containing cond. prob. only when defined, and only consider homogeneous MC.

1.5 Def.

For a MC $(X_n)$, the matrix $P = (P_{ij}) \in \{0, 1\}^{E \times E}$ with $P_{ij} = P(X_{n+1} = j \mid X_n = i)$ is called transition matrix of $(X_n)$.

A function $\pi : E \to [0, 1]$ with $\sum_{i \in E} \pi(i) = 1$ and $\pi(i) = P(X_0 = i)$ is called initial distribution of $(X_n)$. 
1.6 Remarks

$P_{ij}$ is the one-step transition probability from state $i$ to state $j$.

It follows that $P_{ij} \geq 0$ for $i,j \in E$ and $\sum_{j \in E} P_{ij} = 1$, i.e., $P$ is a stochastic matrix.

1.7 Theorem The following statements are equivalent:

1. $(X_n)$ is a MC,
2. if $X_0 \sim \text{PM}(\pi_0), (P_{ij})_{i,j \in E} \in [0,1]^{E \times E} \text{ and } \sum_{j \in E} P_{ij} = 1$, then $P(X_n = i_0 | X_{n-1} = i_1, \ldots, X_1 = i_{n-1}) = \pi_{i_0} \pi_{i_1} \ldots \pi_{i_{n-1}}$

1.8 Remarks

A MC can be defined by giving a state space, a corresponding stochastic matrix, and an initial distribution.

If no initial distribution is specified, the transition matrix describes a family of MCs.

1.9 Exercise Describe repression of a gene:

$E = \{0,1\}$, $X_n = 0$ - gene free at time $n$

$X_n = 1$ - gene repressed at time $n$

Assumptions:
1. gene free at time $n$, then gene repressed at time $n+1$ with prob. $p > 0$
2. gene repressed at time $n$, then gene free at time $n+1$ with prob. $q > 0$

$\Rightarrow P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$

1.10 Definition Given the transition matrix $P \in [0,1]^{E \times E}$ of a MC $(X_n)$ with state space $E$, the directed labeled graph $G$ with vertex set $E$ and edges (i.e., $P_{ij}$), $i,j \in E$, $P_{ij} > 0$ is called the transition graph of $(X_n)$.

1.11 Exercise Random walk on $E = \mathbb{N}$.

Choose $P_i \in (0,1)$ for all $i \in \mathbb{N}$.

Diagram:

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1 --- P1 --> 2 --- P2 --> 3 --- P3 --> 4 --- P4 --> ...
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1-P2 1-P3 1-P4
Canonical representation

2.1 Theorem
Let \((Z_k)_{k \in \mathbb{N}}\) be a sequence of independent and identically distributed (i.i.d.) random variables \(Z_k : \Omega \to D\) (with \((D, \mathcal{D})\) some measurable space), and let \(X_0 : \Omega \to E\) be a random variable independent of \(Z_1, Z_2, \ldots\). Consider a function \(f : E \times D \to E\). Then the recurrence equation

\[
X_{k+1} = f(X_k, Z_{k+1})
\]

defines a homogeneous Markov chain \((X_k)_{k \in \mathbb{N}}\) on state space \(E\).

2.2 Remark
For the transition probabilities holds \(p_{ij} = P(X_1 = j \mid X_0 = i) = P(f(i, Z_k) = j)\).

2.3 Example Random walk on \(Z\)
\(X_0 : \Omega \to \mathbb{Z}\) independent of i.i.d. random variables \(Z_1, Z_2, \ldots : \Omega \to \{-1, 1\}\) with \(P(Z_k = 1) = q, P(Z_k = -1) = 1 - q\) for some \(q \in (0, 1)\).

MC \((X_n)\) defined by \(X_{n+1} = f(X_n, Z_{n+1}) = X_n + Z_{n+1}, f : \mathbb{Z} \times \{-1, 1\} \to \mathbb{Z}\)

2.4 Remark
1. Given \((X_n)\), \(P = (p_{ij})\) i.e. one can always define

   - \((Z_k)\) sequence of i.i.d. random variables uniformly distributed on \([0, 1]\)
   - \(f : E \times [0, 1] \to E\) with \(f(i, z) = \lfloor k \frac{b}{z} \rfloor \text{ if } \sum_{j=1}^{k} p_{ij} \leq z \leq \sum_{j=1}^{k+1} p_{ij}\)
     (i.e. \(f\) not adopting value \(k\) if \(\sum_{j=1}^{k-1} p_{ij} = \sum_{j=1}^{k} p_{ij}\), i.e. \(P_{ik} = 0\)).

2. Simulation of \((X_n)\): Obtain realisation \((x_0, x_1, x_2, \ldots)\)

   - choose \(x_0 \in E\) randomly (acc. to distribution of \(X_0\))
   - choose \(y_1, y_2, \ldots \in D\) according to \(Z_1\) (used representatively for all \(Z_k\))
   - define \(x_{k+1} = f(x_k, y_{k+1})\)