Reverse Engineering Algorithms

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Review

- Model, $M = (I, K)$. (IG $I$ has no multiple edges).
- ASTG, $T = (X, S)$, the dynamics generated by a model $M$.
- Proposition: 3 $u$-row types.
- Lemma: isomorphic groups of $u$-rows.
Extremal state and extremal row

**Definition**

(Lorenz2011) A state \( x = (x_u)_{u \in V} \) is called an extremal state, if \( x_u \in \{0, \max_u\} \), for all \( u \in V \).

A \( u \)-row \( \tau^u = (x^0, \ldots, x^{\max_u}) \) is extremal, if both \( x^0 \) and \( x^{\max_u} \) are extremal states.
Lemma

(Lorenz2011) Given a component $u \in V$ and a resource $\omega \subseteq \text{Pre}(u)$, there always exists an extremal state $x \in X$ such that $\text{Res}_u(x) = \omega$.

Proof.

For all $v \in V$, define

$$x_v := \begin{cases} 
0 & v \notin \text{Pre}(u) \\
0 & \varepsilon(v, u) = + \land v \notin \omega \\
\max_v \varepsilon(v, u) = + \land v \in \omega \\
\max_v \varepsilon(v, u) = - \land v \notin \omega \\
0 & \varepsilon(v, u) = - \land v \in \omega 
\end{cases} \quad (2.1)$$

Then by construction, $\text{Res}_u(x) = \omega$. $\square$

Extremal state of $v_1$

which has resource $\{v_1\}$

So that, $x = (x_{v_1}, x_{v_2}) =$?
Isomorphic $u$-rows with two states.

**Lemma**

(Lorenz 2013) Let $x, y \in X$ such that there exists a component $u \in V$ with $\text{Res}_u(x) \setminus u = \text{Res}_u(y) \setminus u$. Then the $u$-row $(x^0, \ldots, x^{\max u})$ containing $x$ is isomorphic to the $u$-row $(y^0, \ldots, y^{\max u})$ containing $y$. 

![Diagram](image-url)
Theorem

(Lorenz2013) For any model $M = (I, K)$, the state transition graph $T_M$ is uniquely determined by $I$ and the extremal rows of $T_M$.

Proof.

(Lorenz2013) For any $u$-row $(x^0, \cdots, x^{max_u})$, one can always find an extremal state $y$ with $Res_u(x^0) = Res_u(y)$, according to Lemma 2. Quite directly, from Lemma of isomorphic $u$-rows with two states, the extremal row including $y$ is isomorphic to the $u$-row $(x^0, \cdots, x^{max_u})$. □
Figure: The interaction graph and the extremal rows of an ASTG uniquely determine the complete ASTG. \( \vartheta(v_1, v_2) = 1 \) infers that, \( \tau_{v_2}^{v_1} \) is isomorphic with \( \tau_{v_1}^{v_1} \). Similarly, \( \vartheta(v_2, v_1) = 1 \) infers that, \( \tau_{v_1}^{v_1} \) is isomorphic with \( \tau_{v_1}^{v_1} \).
Figure: (a) IG $I$. (b) Logical parameter function $K$. (c) Corresponding ASTG $T$. (d) Alternative logical parameter function $K'$. $M_1 = (I, K)$ and $M_2 = (I, K')$ are two isomorphic models generating the same ASTG $T$, where $K$ satisfies the Snoussi-condition but $K'$ does not.
Definition

(Equivalent models) Let $M_1 = (I_1, K_1)$ and $M_2 = (I_2, K_2)$ be two models, where $I_1 = (V, E_1, \varepsilon_1, \vartheta_1, \max)$ and $I_2 = (V, E_2, \varepsilon_2, \vartheta_2, \max)$. $M_1$ and $M_2$ are equivalent if

$$\delta_{M_1}(u, x) = \delta_{M_2}(u, x), \forall x \in X, \forall u \in V.$$ 

(Let us see an example for this definition and the following Lemma.)
Lemma

(Lorenz2011) Given a model $M^1 = (I^1, K^1)$ with $I^1 = (V, E, \varepsilon^1, \vartheta^1, \max)$.

a) For IG $I^2 = (V, E, \varepsilon^2, \vartheta, \max)$, a logical parameter function $K^2$ is defined as, for all $u \in V$ and $\omega \in \text{Pre}(u) \mid I^1$

$$K^2(u, \omega) = K^1(u, \omega') \quad \omega' := \omega \Delta \{v \in \text{Pre}(u) \mid \varepsilon^1(v, u) \neq \varepsilon^2(v, u)\}$$

then, the model $M^2 = (I^2, K^2)$ defines the same ASTG as $M^1$.

b) For an IG $I^3 = (V, E^3 = E \sqcup E', \varepsilon^3, \vartheta^3, \max)$, for all $u \in V$, one can define its predecessors in the following way:

$\text{Pre}_E(u) := \{(v, u) \mid (v, u) \in E\}$, $\text{Pre}_{E'}(u) := \{(v, u) \mid (v, u) \in E'\}$, $\text{Pre}_{E^3}(u) := \text{Pre}_E(u) \cup \text{Pre}_{E'}(u)$. If the thresholds on those interactions from $E$, $\vartheta^3 \mid E$ is identical with $\vartheta$, i.e., $\vartheta^3 \mid E \equiv \vartheta$, then one can define a $K^3$ as follows: $\forall u \in V \quad \forall \omega \subseteq \text{Pre}_{E^3}(u)$:

$$\omega' := \omega \Delta \{v \in \text{Pre}_E \mid \varepsilon^1(v, u) \neq \varepsilon^3(v, u)\}, \quad K^3(u, \omega) := K^1(u, \omega')$$

$M^3 := (I^3, K^3)$ defines the same ASTG as $M^1$. Moreover, $K^3$ is chosen that no edges in $E'$ is visible.
(a) $I^1$.  
(b) $I^2$.  
(c) $I^3$.  

(d) $K^1$  
(e) $K^2$  
(f) $K^3$  

Figure: (a), (d) $M^1 = (I^1, K^1)$, (b), (e) $M^2 = (I^2, K^2)$, (c), (f) $M^3 = (I^3, K^3)$.  

Figure: The ASTG $T$ of the three models in Figure 3.