Basic solutions

- $Ax \leq b$, $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = n$.
- $M = \{1, \ldots, m\}$ row indices, $N = \{1, \ldots, n\}$ column indices
- For $I \subseteq M, J \subseteq N$ let $A_{IJ}$ denote the submatrix of $A$ defined by the rows in $I$ and the columns in $J$.
- $I \subseteq M, |I| = n$ is called a basis of $A$ iff $A_{I\ast} = A_{IN}$ is non-singular.
- If $x = A_{I\ast}^{-1}b_i$ satisfies $Ax \leq b$, then $x$ called a basic feasible solution and $I$ is called a feasible basis.

Algebraic characterization of vertices

Theorem

Given the non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $\text{rank}(A) = n$, a vector $v \in \mathbb{R}^n$ is a vertex of $P$ if and only if it is a basic feasible solution of $Ax \leq b$, for some basis $I$ of $A$.

For any $c \in \mathbb{R}^n$, either the maximum value of $z = c^T x$ for $x \in P$ is attained at a vertex of $P$ or $z$ is unbounded on $P$.

Corollary

$P$ has at least one and at most finitely many vertices.

Remark

In general, a vertex may be defined by several bases.

Simplex Algorithm: Algebraic version

- Suppose $\text{rank}(A) = n$ (otherwise apply Gaussian elimination).
- Suppose $I$ is a feasible basis with corresponding vertex $v = A_{I\ast}^{-1}b_i$.
- Compute $u^T = c^T A_{I\ast}^{-1}$ (vector of $n$ components indexed by $I$).
- If $u \geq 0$, then $v$ is an optimal solution, because for each feasible solution $x$
  $$c^T x = u^T A_{I\ast} x \leq u^T b_i = u^T A_{I\ast} v = c^T v.$$  
- If $u \not\geq 0$, choose $i \in I$ such that $u_i < 0$ and define the direction $d = -A_{I\ast}^{-1} e_i$, where $e_i$ is the $i$-th unit basis vector in $\mathbb{R}^I$.
- Next increase the objective function value by going from $v$ in direction $d$, while maintaining feasibility.

Simplex Algorithm: Algebraic version (2)

1. If $Ad \leq 0$, the largest $\lambda \geq 0$ for which $v + \lambda d$ is still feasible is
   $$\lambda^* = \min \left\{ \frac{b_l - A_{I\ast}v}{A_{I\ast}d} \mid l \in \{1, \ldots, m\}, A_{I\ast}d > 0 \right\}.$$  
   Let this minimum be attained at index $k$. Then $k \not\in I$ because $A_{I\ast}d = -e_i \leq 0$.
   Define $I' = (I \setminus \{i\}) \cup \{k\}$, which corresponds to the vertex $v + \lambda^* d$.
   Replace $I$ by $I'$ and repeat the iteration.
2. If $Ad \leq 0$, then $v + \lambda d$ is feasible, for all $\lambda \geq 0$. Moreover,

$$c^T d = -c^T A_{i+1}^{-1} e_i = -u^T e_i = -u_i > 0.$$ 

Thus the objective function can be increased along $d$ to infinity and the problem is unbounded.

**Termination and complexity**

- The method terminates if the indices $i$ and $k$ are chosen in the right way (such choices are called pivoting rules).

- Following the rule of Bland, one can choose the smallest $i$ such that $u_i < 0$ and the smallest $k$ attaining the minimum in (PIV).

- For most known pivoting rules, sequences of examples have been constructed such that the number of iterations is exponential in $m+n$ (e.g. Klee-Minty cubes).

- Although no pivoting rule is known to yield a polynomial time algorithm, the Simplex method turns out to work very well in practice.

**Simplex : Phase I**

- In order to find an initial feasible basis, consider the auxiliary linear program

$$\max \{ y \mid Ax - by \leq 0, \ -y \leq 0, \ y \leq 1 \}, \quad \text{(Aux)}$$

where $y$ is a new variable.

- Given an arbitrary basis $K$ of $A$, obtain a feasible basis $I$ for (Aux) by choosing $I = K \cup \{ m+1 \}$. The corresponding basic feasible solution is 0.

- Apply the Simplex method to (Aux). If the optimum value is 0, then (LP) is infeasible. Otherwise, the optimum value has to be 1.

- If $I'$ is the final feasible basis of (Aux), then $K' = I' \setminus \{ m+2 \}$ can be used as an initial feasible basis for (LP).

**Application: Metabolic networks**